

EQUALITY OF BULK AND EDGE HALL CONDUCTANCES FOR CONTINUOUS MAGNETIC RANDOM SCHRÖDINGER OPERATORS

AMAL TAARABT

ABSTRACT. In this note, we prove the equality of the quantum bulk and the edge Hall conductances in mobility edges and in presence of disorder. The bulk and edge perturbations can be either of electric or magnetic nature. The edge conductance is regularized in a suitable way to enable the Fermi level to lie in a region of localized states.

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1. INTRODUCTION

A large literature has emerged a few years only after the discovery of the integer quantum Hall effect [KDP]. Laughlin followed by Halperin, argued that the occurrence of the *plateaux* is due to the localization phenomenon [Hal, L]. The presence of impurities is indeed imperative in order to observe the quantum Hall effect [B, BESB]. In a disorder media, the energy spectrum consists in bands of extended

states separated by energy regions of localized states or energy gaps [BESB, ?]. If the Fermi energy lies in the extremities of these bands, where localization holds, the Hall conductance is constant. The quantum Hall conductance jumps from one integer value to another near the centers until to find a new localization region. Halperin formulates the existence of the edge currents in the Hall systems [Hal]. Indeed, the electrons flowing along the edge of the system rebound and induce then currents which are quantized through the edge conductance. He established that these conductances are *a priori* equal.

The mathematical study of the quantization of Hall conductances has been first developed in parallel. While Bellissard and followers [B, BESB, ASS, BoGKS, GKS1, GKS2] were interested in the Hall conductance, its topological nature, its quantization, and its derivation from a Kubo formula of the quantum Hall effect which is a part of the theory of noncommutative geometry, [CG, CGH, DGR1, KSB, KRSB1, KRSB2] are rather devoted to the edge currents and their quantization. These simultaneous quantizations highlight the equality of the edge and bulk conductances that [EG, EGS] showed by derivation in the discrete case. Elbau and Graf showed that the bulk and the edge conductances matches and are equal under a gap condition [EG]. It was later improved in [EGS] for energy intervals lying in localization region of the bulk Hamiltonians. Our goal in this paper is to prove that equality within the context of random magnetic Schrödinger operators in the continuum and in presence of electric or magnetic wall.

A great interest has been focused in the recent years on the study of random magnetic fields and their localisation properties [AHK, BSK, CH, CHKR, DGR2, GhHK, W]. To describe the bulk in our model, we consider electric and magnetic random perturbations of the free Landau Hamiltonian of Anderson type. The associated Hall conductance is stemmed from the Kubo formula. We then introduce a confining wall, that will be sent to infinity. The models that we deal with are purely electric or purely magnetic (wall and random perturbation). However, we could also consider variants with an electric random operator and a magnetic wall and vice-versa. We define the associated operators and edge conductance.

It is important to emphasize that a localization regime requires a regularization of the usual edge conductance. These regularizations are intended to cancel the contributions of states living away from the edge that might generate extra currents and to restore the trace class property which could be destroyed. We shall make use a regularization introduced in [EGS], and establish the equality of the bulk and edge Hall conductances by deriving one from the other, and not by separate quantization as in [CG].

The paper is organized as follow. In section 2, we introduce our bulk models and formulate the localization assumption. The section 3 is devoted to the description of models with electric or magnetic walls and the associated edge conductance. The section 4, we state our main result and we sketch the strategy of its proof. In section 5, we provide the full proofs of the key steps described in section 4. Appendix A and Appendix B contain some technical tools and trace-class properties.

2. BULK MODELS

We consider the Landau Hamiltonian

$$H_B = (-i\nabla - \mathcal{A}_0)^2 \quad \text{with} \quad \mathcal{A}_0(x_1, x_2) = \frac{B}{2}(-x_2, x_1), \quad (2.1)$$

where \mathcal{A}_0 is the vector potential generating a magnetic field with strength $B > 0$ constant. We shall consider electric and magnetic perturbations V and A of H_B and we set

$$H_B(A, V) := (-i\nabla - \mathcal{A}_0 - A)^2 + V.$$

We recall the Leinfelder-Simader conditions (LS) for an operator of the form

$$H(A, V) := (-i\nabla - A)^2 + V. \quad (2.2)$$

We say that the magnetic potential A and the electric potential V satisfy the Leinfelder-Simader conditions if

- (1) $A \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ avec $\text{div } A \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R})$.
- (2) $V(x) = V_+(x) - V_-(x)$ with $V_{\pm} \geq 0$, $V_{\pm} \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R})$ and V_- relatively bounded with respect to $-\Delta$ with relative bound < 1 such that there exist $0 \leq \alpha < 1$ and $\beta \geq 0$ independent of ω so that for all $\psi \in \mathcal{D}(\Delta)$, we have

$$\|V_- \psi\| \leq \alpha \|\Delta \psi\| + \beta \|\psi\|.$$

Under these conditions, the operator $H(A, V)$ in (2.2) is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^2)$ [LS].

In this work, we are interested in random perturbations of H_B of electric and magnetic nature.

2.1. Electric model. We consider the random Landau Hamiltonian

$$H_\omega^E = H_B + V_\omega \quad \text{on } L^2(\mathbb{R}^2), \quad (2.3)$$

with V_ω a random potential of Anderson-type

$$V_\omega := \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma u(\cdot - \gamma), \quad (2.4)$$

where $\omega = (\omega_\gamma)_{\gamma \in \mathbb{Z}^2}$ is a family of independent and identically distributed (iid) random variables and the single site potential u is a nonnegative bounded measurable function on \mathbb{R}^2 with compact support such that $-M_1 \leq V_\omega \leq M_2$ with $0 \leq M_1, M_2 < \infty$. We assume that the family $(\omega_\gamma)_\gamma$ has a common non-degenerate probability distribution μ with bounded density ρ . We write (Ω, \mathbb{P}) for the underlying probability space and \mathbb{E} for the corresponding expectation.

Using the magnetic translation U_α defined by

$$(U_\alpha \psi)(x) = e^{-i\frac{B}{2}\alpha \wedge x} \psi(x - \alpha) \quad \text{for } \alpha \in \mathbb{R}^2, \quad (2.5)$$

where $\alpha \wedge x = \alpha_1 x_2 - \alpha_2 x_1$, it follows that the random operator H_ω is \mathbb{Z}^2 -ergodic and is essentially self-adjoint with core $\mathcal{C}_c^\infty(\mathbb{R}^2)$. Hence, it follows from [CL] that H_ω^E has a nonrandom spectrum and there exists a deterministic set $\Sigma_E \subset \mathbb{R}$ such that $\sigma(H_\omega^E) = \Sigma_E$ with probability one.

The spectrum of the free Landau Hamiltonian H_B given in (2.1) consists in a sequence of infinitely degenerated eigenvalues, called Landau levels

$$B_n = (2n - 1)B, \quad n = 1, 2, \dots \quad (2.6)$$

with the convention $B_0 = -\infty$. And we have

$$\Sigma_E \subset \bigcup_n [B_n - M_1, B_n + M_2], \quad (2.7)$$

and there is no overlap provided that the open gap condition $M_1 + M_2 < 2B$ is fulfilled.

Remark 2.1. *The assumption on the bounded density ρ of the random variables is made in order to cover models that are known to exhibit dynamical localisation.*

2.2. Pure magnetic model. Let \mathcal{A}_ω be a random vector potential of the form

$$\mathcal{A}_\omega = \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma u(\cdot - \gamma),$$

satisfying the Leinfelder-Simader conditions [LS] almost surely. The single site functions $u = (u_1, u_2) \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$ are compactly supported and the random variables $\omega = (\omega_\gamma)_{\gamma \in \mathbb{Z}^2}$ are independant and identically distributed (iid) with common probability distribution. The probability space is again denoted by (Ω, \mathbb{P}) . We consider the magnetic random operator

$$H_\omega^M = (-i\nabla - \mathcal{A}_0 - \mathcal{A}_\omega)^2 \quad \text{on } L^2(\mathbb{R}^2), \quad (2.8)$$

which is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^2)$ and uniformly bounded from below for \mathbb{P} -a.e ω . By ergodicity, we denote its spectrum $\sigma(H_\omega^M)$ by Σ_M .

The operators H_ω^E and H_ω^M are essentially self-adjoint and bounded from below: there exists $\Theta_\bullet \geq 1$ such that $H_\omega^\bullet + \Theta_\bullet \geq 1$ [BoGKS] and where $\bullet = E, M$. For simplification and since our analysis remains essentially the same for both models, we may omit E and M from the notations and write H_ω to denote H_ω^E and H_ω^M . Nevertheless, when needed, we will specify the case we deal with.

2.3. Localisation. For $m > 0$ and $\zeta \in (0, 1]$ given, we introduce the random (m, ζ) -subexponential moment at time t for the time evolution, initially localized around the origin and localized in energy by the function $\mathcal{X} \in \mathcal{C}_{c,+}^\infty(I)$,

$$M_\omega(m, \zeta, \mathcal{X}, t) := \left\| e^{\frac{m}{2}|X|^\zeta} e^{-itH_\omega} \mathcal{X}(H_\omega) \chi_0 \right\|_2^2. \quad (2.9)$$

We define its time average expectation as

$$\mathcal{M}(m, \zeta, \mathcal{X}, T) := \frac{1}{T} \int_0^\infty e^{-t/T} \mathbb{E}\{M_\omega(m, \zeta, \mathcal{X}, t)\} dt. \quad (2.10)$$

Given an energy $E \in \mathbb{R}$, we consider the Fermi projector $P_\omega^{(E)} = \chi_{(-\infty, E]}(H_\omega)$, the spectral projection of H_ω onto energies below E .

Definition 2.2.

(Loc) *We say that the operator H_ω exhibits localization in an open interval I if there exist $m > 0, \zeta \in (0, 1)$ so that for any $\mathcal{X} \in \mathcal{C}_{c,+}^\infty(I)$, we have*

$$\sup_T \mathcal{M}(m, \zeta, \mathcal{X}, T) < \infty. \quad (2.11)$$

We denote by Σ_{loc} the region of localization

$$\Sigma_{loc} := \{E \in \mathbb{R} : H_\omega \text{ exhibits localization in a neighborhood of } E\}. \quad (2.12)$$

(DFP) *The Fermi projection $P_\omega^{(E)}$ exhibits sub-exponential decay if the Fermi energy $E \in \Sigma_{loc}$ and if there exist $m > 0, \zeta \in (0, 1)$ such that we have*

$$\mathbb{E} \left\{ \left\| \chi_x P_\omega^{(E)} \chi_y \right\|_2^2 \right\} \leq C_{m, \zeta, B, E} e^{-m|x-y|^\zeta} \quad \text{for all } x, y \in \mathbb{Z}^2, \quad (2.13)$$

where the constant $C_{m,\zeta,B,E}$ is locally bounded in E . As a consequence, for any $\epsilon > 0$ and \mathbb{P} -a.e ω , we have

$$\left\| \chi_x P_\omega^{(E)} \chi_y \right\|_2 \leq C_{\omega,m,\zeta,\epsilon,B,E} e^{\epsilon|x|^\zeta} e^{-m|x-y|^\zeta} \quad \text{for all } x, y \in \mathbb{Z}^2. \quad (2.14)$$

The existence of the region of localization (2.12) has been proven in [CH, GK2, DGR2]. Moreover, it corresponds to the region where the bootstrap multiscale analysis (MSA) can be performed [GK1, GK2]. The magnetic models are treated in [DGR2, GhHK]. The **(DFP)** property and (2.14) play an important role in the study and the definition of the Hall conductance.

2.4. Hall conductance. Consider a smooth characteristic function $\Lambda(s)$ which is equal to 1 for $s \leq -\frac{1}{2}$ and 0 for $s \geq \frac{1}{2}$ such that $\text{supp } \Lambda' \subset (-\frac{1}{2}, \frac{1}{2})$. Let Λ_j denotes the multiplication operator by the function $\Lambda_j(x) = \Lambda(x_j)$ for $j = 1, 2$.

Definition 2.3. The Hall conductance at a Fermi energy E is defined by

$$\sigma_{\text{Hall}}(B, \omega, E) := -i \text{tr} \left[P_\omega^{(E)} \Lambda_2 P_\omega^{(E)}, P_\omega^{(E)} \Lambda_1 P_\omega^{(E)} \right]. \quad (2.15)$$

In view of (2.13), it is well defined in Σ_{loc} (see [GKS1]). The ergodicity property implies that (2.15) is a nonrandom quantity in the sense that for \mathbb{P} -a.e ω ,

$$\sigma_{\text{Hall}}(B, E) := \mathbb{E} \{ \sigma_{\text{Hall}}(B, \omega, E) \} = \sigma_{\text{Hall}}(B, \omega, E). \quad (2.16)$$

Notice that the operators $P_\omega^{(E)} \Lambda_2 P_\omega^{(E)}$ and $P_\omega^{(E)} \Lambda_1 P_\omega^{(E)}$ in (2.15) are not separately trace class otherwise the commutator would be zero.

The Hall conductance $\sigma_{\text{Hall}}(B, E)$ is known to be constant in Σ_{loc} [?]. This corresponds to the occurrence of the well-known plateaux in the QHE.

Remark 2.4. There are alternative definitions to (2.15), namely

$$-i \text{tr} P_\omega^{(E)} \left[\left[P_\omega^{(E)}, \Lambda_2 \right], \left[P_\omega^{(E)}, \Lambda_1 \right] \right]. \quad (2.17)$$

Note that the operator $\left[P_\omega^{(E)}, \Lambda_2 \right] \left[P_\omega^{(E)}, \Lambda_1 \right]$ in (2.17) is morally supported near the origin. One can also consider

$$-i \text{tr} \{ \chi_0 P_\omega^{(E)} \left[\left[P_\omega^{(E)}, X_2 \right], \left[P_\omega^{(E)}, X_1 \right] \right] \chi_0 \}, \quad (2.18)$$

where X_i is the multiplication operator by the coordinate x_i for $i = 1, 2$.

3. MODELS WITH WALLS

In this note, we are interested in soft walls of magnetic or electric nature.

3.1. Electric edge. Let $U \in \mathcal{C}^\infty(\mathbb{R}^2)$ be an x_2 -invariant decreasing function such that

$$\begin{cases} \lim_{x_1 \rightarrow -\infty} U(x_1) = U_- < \infty, \\ U(x_1) = 0 \quad \text{for } x_1 \geq 0. \end{cases} \quad (3.1)$$

We should consider U_- sufficiently large compared to the energy zone where we work. The electric edge operator is giving by

$$H_{\omega,a}^E = H_B + U_a + V_\omega, \quad (3.2)$$

where $a > 0$ and U_a is the multiplication by the function $U_a(x_1) = U(x_1 + a)$ which translate the wall and placing it at $x_1 = -a$. It is a soft and left confining wall in the sense that the particle remains trapped and confined on the right side of the plane.

3.2. Magnetic edge. Let $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ be a vector potential generating the magnetic field $\mathcal{B} : \mathbb{R}^2 \rightarrow \mathbb{R}$, i.e,

$$\nabla \wedge \mathcal{A}(x) = \mathcal{B}(x), \text{ for } x = (x_1, x_2) \in \mathbb{R}^2, \quad (3.3)$$

where \mathcal{B} is a smooth decreasing x_2 -invariant function so that

$$\begin{cases} \lim_{x_1 \rightarrow -\infty} \mathcal{B}(x_1) = B_- < \infty, \\ \mathcal{B}(x_1) = 0 \quad \text{for } x_1 \geq 1. \end{cases} \quad (3.4)$$

Once again, like the electric case above, we translate this wall with a parameter $a > 0$ so that

$$\frac{\partial \mathcal{A}_2}{\partial x_1}(x_1 + a, x_2) - \frac{\partial \mathcal{A}_1}{\partial x_2}(x_1 + a, x_2) = \mathcal{B}(x_1 + a) := \mathcal{B}_a(x_1). \quad (3.5)$$

In that case, the Magnetic edge operator is

$$H_{\omega,a}^M = (-i\nabla - \mathcal{A}_0 - \mathcal{A}_a - \mathcal{A}_\omega)^2. \quad (3.6)$$

If we set $\mathcal{A}_a^{\text{lw}} = \mathcal{A}_0 + \mathcal{A}_a$, we obtain the so-called Iwatsuka magnetic field with limits in $+\infty$ and $-\infty$ given by $B + B_-$ and B respectively [CFKS, DGR1, E, I].

In view of the gauge invariance for magnetic operators, one can choose a suitable transformation and simplify the spectral studies of magnetic operators of the form $(-i\nabla - \mathcal{A})^2$. Let us consider the Landau gauge and take $\mathcal{A} = (0, \mathcal{A}_2)$ where $\mathcal{A}_2 = \beta(x_1) := \int_0^{x_1} \mathcal{B}(s)ds$. The invariance in x_2 -direction allows the performance of the partial Fourier transform with respect to the variable x_2 . Hence, the operator $H(\mathcal{A})$ can be written as

$$H(\mathcal{A}) = -\frac{\partial^2}{\partial x_1^2} + (-i\frac{\partial}{\partial x_2} - \beta(x_1))^2. \quad (3.7)$$

Then it is unitary equivalent to

$$h(k) := -\frac{d^2}{dx_1^2} + (k - \beta(x_1))^2, \text{ for } k \in \mathbb{R}, \quad (3.8)$$

whose spectrum is discrete [E].

By $H_{\omega,a}$, we mean both $H_{\omega,a}^E$ and $H_{\omega,a}^M$. Notice that the edge operators $H_{\omega,a}$ converge to H_ω in strong resolvent sense. Hence $H_{\omega,a} \rightarrow H_\omega$ in the strong resolvent sense (see appendix B.1). In order to justify this strong convergence, we the resolvent identity and we consider the difference operator

$$\Gamma_{\omega,a}^M = H_{\omega,a}^M - H_\omega^M = -2\mathcal{A}_a \cdot (-i\nabla - \mathcal{A}_0 - \mathcal{A}_\omega) + i \operatorname{div} \mathcal{A}_a + |\mathcal{A}_a|^2, \quad (3.9)$$

and

$$\Gamma_{\omega,a}^E = H_{\omega,a}^E - H_\omega^E = U_a. \quad (3.10)$$

Since the operator $\Gamma_{\omega,a}^\bullet R_{\omega,a}$ is uniformly bounded in a for $\bullet = E, M$ and the compactly supported functions are dense in \mathcal{H} , it suffices to verify this strong convergence in $\mathcal{C}_0^\infty(\mathbb{R}^2)$. We consider a test function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ leaving far apart from the wall such that $\operatorname{supp} \phi \cap \operatorname{supp} \mathcal{B}_a = \emptyset$, according to [T].

3.3. Edge conductance. We start with the definition of switch functions.

Definition 3.1. Let $g : \mathbb{R} \rightarrow [0, 1]$ be a smooth decreasing function. We say that g is a switch function if it has a compactly supported derivative such that $g \equiv 1$ on the left side of $\text{supp } g'$ and $g \equiv 0$ on the right one.

We say that g is a switch function of an interval I if $\text{supp } g' \subset I$.

Heuristically, the current along $x_1 = -a$ and in direction x_2 induced by states with energy support in an interval I , is given by

$$J(I) = \text{tr}(E_I(H_{\omega,a})i[H_{\omega,a}, \Lambda_2]),$$

where $E_I(H_{\omega,a})$ is the spectral projection of $H_{\omega,a}$ on I . The edge conductance is then the ratio

$$\sigma_e(\omega, I) = \frac{J(I)}{|I|} \approx -i \text{tr}(g'(H_{\omega,a})[H_{\omega,a}, \Lambda_2]),$$

since

$$-g'(H_{\omega,a}) \approx \frac{E_I(H_{\omega,a})}{|I|},$$

where I lies in a spectral gap of H_ω . However, it is more relevant for physical interest, to consider the case where \mathcal{I} falls into Σ_{loc} , region of localized states so that $I \cap \Sigma_{loc} \neq \emptyset$. In fact, such states might generate spurious currents that we have to cancel. In order to treat this case, a regularization of the edge conductance is required. Some regularizations have been proposed in [CG, CGH] and [EGS]. The second candidate of [EGS] is a time-average regularization where they considered the Heisenberg evolution of Λ_1 and time-averaged the final expression. It is the regularization that we shall consider.

Definition 3.2. Let $I \subset (B_n, B_{n+1}) \cap \Sigma_{loc}$ be a given interval for some n . Let g be a decreasing switch function of I . The regularized edge conductance of H_ω in I is defined as

$$\sigma_{e,\omega}^{\text{reg}} := \lim_{T \rightarrow \infty} \lim_{a \rightarrow \infty} \frac{1}{T} \int_0^T -i \text{tr } g'(H_{\omega,a})[H_{\omega,a}, \Lambda_2] \Lambda_{1,a}^\omega(t) dt, \quad (3.11)$$

whenever the limits exist and where $\Lambda_{1,a}^\omega(t) := e^{itH_{\omega,a}} \Lambda_1 e^{-itH_{\omega,a}}$.

Since the operator $g'(H_{\omega,a})[H_{\omega,a}, \Lambda_2] \Lambda_{1,a}^\omega(t)$ is bounded, we only have to verify that the trace in (3.11) is well defined and that such limits exist. The idea relies on the fact that far from the edge, the dynamic of $\Lambda_{1,a}^\omega$ approaches that of $e^{itH_\omega} \Lambda_1 e^{-itH_\omega}$.

Remark 3.3. Notice that both definitions (2.15) and (3.11) do not depend either on g as long as $\text{supp } g' \subset I$ nor on Λ_j for $j = 1, 2$.

4. MAIN RESULT

4.1. Bulk-Edge equality. Our main result states that in the localization zone (2.12) of the Bulk operator H_ω and in presence of a confining edge, the Hall and edge conductances match and they are equal. This result extends the main result of [EGS] to the continuous setting and to purely random magnetic Schrödinger operators.

Theorem 4.1. *Let $I \subset (B_n, B_{n+1}) \cap \Sigma_{loc}$ be an interval for some $n \in \mathbb{N}$ such that $U_-, B_- > \sup I$. Then for any switch function g of I and any $E \in \text{supp } g'$, the edge conductance (3.11) is well defined and we have*

$$\sigma_{e,\omega}^{\text{reg}} = \sigma_{\text{Hall}}(B, \omega, E) \quad \text{for } \mathbb{P} - \text{a.e. } \omega. \quad (4.1)$$

4.2. Strategy of the proof. Throughout the next sections, we fix $\omega \in \Omega$ and we let I to be an interval such that $I \subset (B_n, B_{n+1}) \cap \Sigma_{loc}$ for some $n \in \mathbb{N}$ given. Let g be a switch function of I . The core of the proof of Theorem 4.1 is based on some intermediate steps that we shall outline below.

First, we compare the operators $g'(H_{\omega,a}) [H_{\omega,a}, \Lambda_2]$ and $[g(H_{\omega,a}), \Lambda_2]$ using the Helffer-Sjöstrand formulas but applied to the primitive function

$$G(x) := \int_x^\infty g(s) ds.$$

We thus have

$$g(H_{\omega,a}) = -\frac{1}{2\pi} \int \bar{\partial} \tilde{G}(z) R_{\omega,a}^2(z) \, dudv \quad (4.2)$$

and

$$g'(H_{\omega,a}) = \frac{1}{\pi} \int_{\mathbb{R}^2} \bar{\partial} \tilde{G}(z) R_{\omega,a}^3(z) \, dudv, \quad (4.3)$$

where $R_{\omega,a}(z) = (H_{\omega,a} - z)^{-1}$ and $z = u + iv$ and \tilde{G} is a quasi-analytic extension of G of order k for $k = 1, 2, \dots$ [D]. Next, we use the second order resolvent identity

$$[R_{\omega,a}^2(z), \Lambda_j] = -R_{\omega,a}^2(z) [H_{\omega,a}, \Lambda_j] R_{\omega,a}(z) - R_{\omega,a}(z) [H_{\omega,a}, \Lambda_j] R_{\omega,a}^2(z) \quad (4.4)$$

to write

$$\begin{aligned} [g(H_{\omega,a}), \Lambda_2] \Lambda_{1,a}^\omega(t) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \bar{\partial} \tilde{G}(z) R_{\omega,a}^2(z) [H_{\omega,a}, \Lambda_2] R_{\omega,a}(z) \Lambda_{1,a}^\omega(t) \, dudv \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{R}^2} \bar{\partial} \tilde{G}(z) R_{\omega,a}(z) [H_{\omega,a}, \Lambda_2] R_{\omega,a}^2(z) \Lambda_{1,a}^\omega(t) \, dudv. \end{aligned} \quad (4.5)$$

We claim that the operators

$$[g(H_{\omega,a}), \Lambda_2] \Lambda_{1,a}^\omega(t) \quad \text{and} \quad g'(H_{\omega,a}) [H_{\omega,a}, \Lambda_2] \Lambda_{1,a}^\omega(t)$$

are both trace class according to Lemma 4.2. Together with (4.3) and the cyclicity of the trace, we have

$$\text{tr } R_{\omega,a}^3(z) [H_{\omega,a}, \Lambda_2] \Lambda_{1,a}^\omega(t) = \frac{1}{2} \text{tr } R_{\omega,a}^2(z) [H_{\omega,a}, \Lambda_2] \Lambda_{1,a}^\omega(t) R_{\omega,a}(z) \quad (4.6)$$

$$+ \frac{1}{2} \text{tr } R_{\omega,a}(z) [H_{\omega,a}, \Lambda_2] \Lambda_{1,a}^\omega(t) R_{\omega,a}^2(z). \quad (4.7)$$

We thus compare (4.5) and (4.7) and obtain an operator $\mathcal{R}_{\omega,a}(t)$ that we call the remainder operator. We thus get

$$\text{tr } g'(H_{\omega,a}) [H_{\omega,a}, \Lambda_2] \Lambda_{1,a}^\omega(t) = \text{tr } [g(H_{\omega,a}), \Lambda_2] \Lambda_{1,a}^\omega(t) + \text{tr } \mathcal{R}_{\omega,a}(t), \quad (4.8)$$

with

$$\begin{aligned}\mathcal{R}_{\omega,a}(t) &= \frac{1}{2\pi} \int \bar{\partial} \tilde{G}(z) R_{\omega,a}^2(z) [H_{\omega,a}, \Lambda_2] R_{\omega,a}(z) [H_{\omega,a}, \Lambda_{1,a}^\omega(t)] R_{\omega,a}(z) dudv \\ &\quad + \frac{1}{2\pi} \int \bar{\partial} \tilde{G}(z) R_{\omega,a}(z) [H_{\omega,a}, \Lambda_2] R_{\omega,a}^2(z) [H_{\omega,a}, \Lambda_{1,a}^\omega(t)] R_{\omega,a}(z) dudv \\ &\quad + \frac{1}{2\pi} \int \bar{\partial} \tilde{G}(z) R_{\omega,a}(z) [H_{\omega,a}, \Lambda_2] R_{\omega,a}(z) [H_{\omega,a}, \Lambda_{1,a}^\omega(t)] R_{\omega,a}^2(z) dudv.\end{aligned}\tag{4.9}$$

We note that we have intentionally applied Helffer-Sjöstrand calculus to the primitive function G in order to get sufficiently high power of the resolvent.

The key steps of the proof of Theorem 4.1 are stated in forthcoming preliminary lemmas. The strategy consists in sending the wall to infinity by taking the limit $a \rightarrow +\infty$ in (4.8). This leads to bulk quantities that we further average in time and analyze.

We start by showing that the key operators we deal with are trace class.

Lemma 4.2. *Let g be a switch function of an open interval I . Then the operators*

- $[g(H_{\omega,a}), \Lambda_2] \Lambda_1$
- $g'(H_{\omega,a}) [H_{\omega,a}, \Lambda_2] \Lambda_1$
- $[g(H_{\omega,a}), \Lambda_2] \Lambda_{1,a}^\omega(t)$
- $g'(H_{\omega,a}) [H_{\omega,a}, \Lambda_2] \Lambda_{1,a}^\omega(t)$

are trace class for all $t \in \mathbb{R}$. Moreover, we have $\text{tr}[g(H_{\omega,a}), \Lambda_2] \Lambda_1 = 0$.

The next lemma highlights the non-contribution of the remainder operator (4.9).

Lemma 4.3. *Let I to be an interval such that $I \subset (B_n, B_{n+1}) \cap \Sigma_{loc}$ for some $n \in \mathbb{N}$ given. Let g a switch function of I . Then*

$$\lim_{T \rightarrow \infty} \lim_{a \rightarrow \infty} \frac{1}{T} \int_0^T \text{tr} \mathcal{R}_{\omega,a}(t) dt = 0. \tag{4.10}$$

We are thus left with the the first term of the r.h.s of (4.8). We rewrite the operator $[g(H_{\omega,a}), \Lambda_2] \Lambda_{1,a}^\omega(t)$ as $[g(H_{\omega,a}), \Lambda_2] (\Lambda_{1,a}^\omega(t) - \Lambda_1)$, since the operator $[g(H_{\omega,a}), \Lambda_2] \Lambda_1$ has zero trace by Lemma 4.2. This does not change the value of the trace but it provides a localization in space in the x_1 -direction.

Lemma 4.4. *Let I to be an interval such that $I \subset (B_n, B_{n+1}) \cap \Sigma_{loc}$ for some $n \in \mathbb{N}$ given. Let g a switch function of I . Then we have*

$$\lim_{a \rightarrow \infty} \text{tr} [g(H_{\omega,a}), \Lambda_2] (\Lambda_{1,a}^\omega(t) - \Lambda_1) = \text{tr} [g(H_\omega), \Lambda_2] (\Lambda_1^\omega(t) - \Lambda_1) \tag{4.11}$$

for all $t \in \mathbb{R}$.

We can deal now with the resulting bulk expression and evaluate their contributions in time-average.

Lemma 4.5. *Let I to be an interval such that $I \subset (B_n, B_{n+1}) \cap \Sigma_{loc}$ for some $n \in \mathbb{N}$ given. Let g a switch function of I . Then we have*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{tr} [g(H_\omega), \Lambda_2] (\Lambda_1^\omega(t) - \Lambda_1) dt = \int g'(E) \text{tr} \Pi_E dE, \tag{4.12}$$

where

$$\Pi_E = P_\omega^{(E)} \Lambda_2 P_\omega^{(E)\perp} \Lambda_1 P_\omega^{(E)} - P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} \Lambda_1 P_\omega^{(E)\perp} \text{ and } P_\omega^{(E)\perp} = 1 - P_\omega^{(E)}. \quad (4.13)$$

We point out how crucial it is to introduce Λ_1 in $[g(H_{\omega,a}), \Lambda_2] \Lambda_{1,a}^\omega(t)$ for it gives a spatial localization in the x_1 -direction by the difference $\Lambda_{1,a}^\omega(t) - \Lambda_1$. This makes the right operator in (4.11) trace class. The proof of Lemma 4.5 actually shows that after having averaged in time, we only keep the term that comes from this added term Λ_1 .

We now return to the Hall conductance (2.15) which is directly connected to Π_E defined in (4.13) thanks to the following lemma.

Lemma 4.6. *Let I be an interval such that one has $I \subset (B_n, B_{n+1}) \cap \Sigma_{loc}$ for some $n \in \mathbb{N}$. Then for any $E \in I$, we have*

$$\begin{aligned} \sigma_{\text{Hall}}(B, \omega, E) &= i \operatorname{tr} (P_\omega^{(E)} \Lambda_2 P_\omega^{(E)\perp} \Lambda_1 P_\omega^{(E)} - P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} \Lambda_1 P_\omega^{(E)\perp}) \\ &= i \operatorname{tr} \Pi_E. \end{aligned} \quad (4.14)$$

Thanks to these preliminary lemmas and thanks to the assumption on g and to the constancy of the Hall conductance in the localization region [?], we thus deduce

$$\sigma_{e,\omega}^{\text{reg}} = \int g'(E) \sigma_{\text{Hall}}(B, \omega, E) \, dE = \sigma_{\text{Hall}}(B, \omega, E), \quad (4.15)$$

for any $E \in I \subset (B_n, B_{n+1}) \cap \Sigma_{loc}$.

5. PROOFS

In this section, we give the details of the proofs and intermediate steps. We start with the trace class property.

5.1. Proof of Lemma 4.2. We first deal with the operator $[g(H_{\omega,a}), \Lambda_2] \Lambda_1$ that we prove to be trace class with zero trace.

5.1.1. Vanishing trace. We first prove the vanishing trace for the pure magnetic model and we pursue with the electric one.

- *Magnetic case.* We proceed as in [CG] and we split the operator

$$[g(H_{\omega,a}^{\text{M}}, \Lambda_2) \Lambda_1$$

in the x_2 -direction so that for an arbitrary $R > 0$, we write it as the sum of

$$(I_R) = [g(H_{\omega,a}^{\text{M}}, \Lambda_2) \Lambda_1 \mathbf{1}_{|x_2| \leq R}, \quad (5.1)$$

and

$$(\Pi_R) = [g(H_{\omega,a}^{\text{M}}, \Lambda_2) \Lambda_1 \mathbf{1}_{|x_2| > R}, \quad (5.2)$$

where $\mathbf{1}_S$ denotes the characteristic function of a subset $S \subset \mathbb{R}^2$. We first treat I_R in (5.1) that we decompose for $r > 0$ as

$$[g(H_{\omega,a}^{\text{M}}, \Lambda_2) \mathbf{1}_{|x_2| \leq R} \mathbf{1}_{-r_0-r-a \leq x_1 \leq 0} + [g(H_{\omega,a}^{\text{M}}, \Lambda_2) \mathbf{1}_{|x_2| \leq R} \mathbf{1}_{x_1 \leq -r_0-r-a}. \quad (5.3)$$

We set $K = \mathbf{1}_{|x_2| \leq R} \mathbf{1}_{-r_0-r-a \leq x_1 \leq 0}$ appearing in the first term of the r.h.s of (5.3). We notice that

$$[g(H_{\omega,a}^{\text{M}}, \Lambda_2) K = [g(H_{\omega,a}^{\text{M}}, K, \Lambda_1)].$$

It is then sufficient to show that $g(H_{\omega,a}^E)K$ is a trace class operator and use the cyclicity of the trace to deduce immediately that

$$\mathrm{tr} [(g(H_{\omega,a}^M), \Lambda_2)] K = \mathrm{tr} [(g(H_{\omega,a}^M)K, \Lambda_2)] = 0.$$

To do this, it follows from the spectral theorem that

$$g(H_{\omega,a}^M) = h(H_{\omega,a}^M) \quad \text{with} \quad h(s) = \chi_{\{s \geq 1-\Theta\}} g(s), \quad (5.4)$$

where χ is a smooth characteristic function. Notice that the function h has compact support (g verifies $\sup(\mathrm{supp} g') \geq 1-\Theta$ otherwise $g(H_{\omega,a}^M) = 0$) and since K has also compact support, we conclude that $h(H_{\omega,a}^M)K = g(H_{\omega,a}^M)K \in \mathcal{T}_1$ [Si, Theorem 4.1]. To prove a similar property for the remaining terms, we introduce a new operator. We let $\tilde{\mathcal{B}}$ to be a new magnetic field such that

$$\tilde{\mathcal{B}}(x_1, x_2) \geq b_0 > \sup \mathcal{I} \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2,$$

and it coincides with \mathcal{B} for $x_1 \leq -r_0$, $r_0 > 0$. The difference magnetic field $\tilde{\mathcal{B}} - \mathcal{B}$ is then supported on $S_{r_0} := \{x_1 \geq -r_0\} \times \mathbb{R}$. It follows from [DGR1, Proposition 4.2] that there exists a magnetic potential $\tilde{\mathcal{A}}$ generating the magnetic field $\tilde{\mathcal{B}} - \mathcal{B}$ and vanishing on $S_{r_0}^c$. Let us consider the auxiliary operator

$$\tilde{H}_{\omega,a}^M := (-i\nabla - \mathcal{A}_0 - \mathcal{A}_a - \tilde{\mathcal{A}}_a - \mathcal{A}_\omega)^2, \quad (5.5)$$

where $\tilde{\mathcal{A}}_a(x)$ means $\tilde{\mathcal{A}}(x_1 + a, x_2)$. Since $\tilde{H}_{\omega,a}^M - \tilde{\mathcal{B}}_a$ is a non-negative operator, it follows (see [E]) that

$$\inf \sigma(\tilde{H}_{\omega,a}^M) \geq \mathrm{Ess} \inf_{x_1 \in \mathbb{R}} \tilde{\mathcal{B}}_a(x_1) \geq b_0.$$

As a consequence, one has $\sigma(\tilde{H}_{\omega,a}^M) \cap \mathcal{I} = \emptyset$ and since $b_0 > \sup I$ then $g(\tilde{H}_{\omega,a}^M) = 0$. We point out the creation of a forbidden zone where the electrons can not penetrate when we introduce such operators $\tilde{H}_{\omega,a}^M$. We first consider the second term of the r.h.s of (5.3), namely

$$[g(H_{\omega,a}^M), \Lambda_2] \mathbf{1}_{|x_2| \leq R} \mathbf{1}_{x_1 \leq -r_0 - r - a} = [g(H_{\omega,a}^M) - g(\tilde{H}_{\omega,a}^M), \Lambda_2] \mathbf{1}_{|x_2| \leq R} \mathbf{1}_{x_1 \leq -r_0 - r - a}. \quad (5.6)$$

By the Helffer-Sjöstrand formula, we have

$$g(H_{\omega,a}^M) - g(\tilde{H}_{\omega,a}^M) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \bar{\partial} \tilde{g}(u + iv) (R_{\omega,a,M} - \tilde{R}_{\omega,a,M}) du dv. \quad (5.7)$$

We thus have to analyze the operator $(R_{\omega,a,M} - \tilde{R}_{\omega,a,M}) \mathbf{1}_{|x_2| \leq R} \mathbf{1}_{x_1 \leq -r_0 - r - a}$. We use commutators to check that

$$\begin{aligned} [R_{\omega,a,M} - \tilde{R}_{\omega,a,M}, \Lambda_2] &= \Lambda_2 R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} - R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \Lambda_2 \\ &= R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} + R_{\omega,a,M} \Lambda_2 \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \\ &\quad + R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} [\tilde{H}_{\omega,a}^M, \Lambda_2] \tilde{R}_{\omega,a,M} - R_{\omega,a,M} \mathcal{W}_{\omega,a} \Lambda_2 \tilde{R}_{\omega,a,M} \\ &= R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} [\tilde{H}_{\omega,a}^M, \Lambda_2] \tilde{R}_{\omega,a,M} + R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \\ &\quad - R_{\omega,a,M} [\mathcal{W}_{\omega,a}, \Lambda_2] \tilde{R}_{\omega,a,M}, \end{aligned}$$

where the first-order operator $\mathcal{W}_{\omega,a}$ is given by

$$\mathcal{W}_{\omega,a} := \tilde{H}_{\omega,a} - H_{\omega,a} = -2\tilde{\mathcal{A}}_a \cdot (-i\nabla - \mathcal{A}_a^{\mathrm{Iw}} - \mathcal{A}_\omega) + i \operatorname{div} \tilde{\mathcal{A}}_a + |\tilde{\mathcal{A}}_a|^2. \quad (5.8)$$

We thus need to control the trace norms of

$$R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \mathbf{1}_{|x_2| \leq R} \mathbf{1}_{x_1 \leq -r_0 - r - a}, \quad (5.9)$$

and

$$R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} [\tilde{H}_{\omega,a}^M, \Lambda_2] \tilde{R}_{\omega,a,M} \mathbf{1}_{|x_2| \leq R} \mathbf{1}_{x_1 \leq -r_0 - r - a}, \quad (5.10)$$

and

$$R_{\omega,a,M} [\mathcal{W}_{\omega,a}, \Lambda_2] \tilde{R}_{\omega,a,M} \mathbf{1}_{|x_2| \leq R} \mathbf{1}_{x_1 \leq -r_0 - r - a}. \quad (5.11)$$

Now, having in mind that the commutator operators

$$[H_{\omega,a}^M, \Lambda_2] = -i(-i\nabla - \mathcal{A}_0 - \mathcal{A}_a - \mathcal{A}_\omega) \cdot \nabla \Lambda_2 - i\nabla \Lambda_2 \cdot (-i\nabla - \mathcal{A}_0 - \mathcal{A}_a - \mathcal{A}_\omega),$$

and

$$[\tilde{H}_{\omega,a}^M, \Lambda_2] = -i(-i\nabla - \mathcal{A}_0 - \mathcal{A}_a - \tilde{\mathcal{A}}_a - \mathcal{A}_\omega) \cdot \nabla \Lambda_2 - i\nabla \Lambda_2 \cdot (-i\nabla - \mathcal{A}_0 - \mathcal{A}_a - \tilde{\mathcal{A}}_a - \mathcal{A}_\omega),$$

are localized on the support of $\nabla \Lambda_2$, we let $\chi_{|x_2| < 1}$ be a smooth characteristic function of $\mathbb{R} \times \{|x_2| < 1\}$ so that we write

$$[H_{\omega,a}^M, \Lambda_2] = [H_{\omega,a}^M, \Lambda_2] \chi_{|x_2| < 1} \quad \text{and} \quad [\tilde{H}_{\omega,a}^M, \Lambda_2] = [\tilde{H}_{\omega,a}^M, \Lambda_2] \chi_{|x_2| < 1}. \quad (5.12)$$

We use unit cubes to decompose $\chi_{|x_2| < 1}$ as

$$\sum_{x=(x_1,0) \in \mathbb{Z}^2} \chi_x,$$

where $(\chi_x)_{x \in \mathbb{Z}^2}$ is a smooth decomposition of unity. We further let

$$\mathcal{W}_{\omega,a} = \sum_{\substack{y_1 \in \mathbb{Z} \cap [-a-r_0, \infty) \\ y_2 \in \mathbb{Z}}} \mathcal{W}_{\omega,a} \chi_y = \sum_{\substack{y_1 \in \mathbb{Z} \cap [-a-r_0, \infty) \\ y_2 \in \mathbb{Z}}} \chi_y \mathcal{W}_{\omega,a}, \quad (5.13)$$

and

$$\mathbf{1}_{|x_2| \leq R} \mathbf{1}_{x_1 \leq -r_0 - r - a} = \sum_{\substack{z_1 \in \mathbb{Z} \cap (\infty, -r_0 - a - r] \\ z_2 \in \mathbb{Z} \cap [-R, R]}} \chi_z. \quad (5.14)$$

To treat (5.9), we write

$$\begin{aligned} (5.9) &= R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] \chi_{|x_2| \leq 1} R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \mathbf{1}_{|x_2| \leq R} \mathbf{1}_{x_1 \leq -r_0 - r - a} \\ &= \sum_{x,y,z \in \mathcal{S}_1} R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} \chi_x \chi_y \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_z \end{aligned} \quad (5.15)$$

$$+ \sum_{x,y,z \in \mathcal{S}_1} R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} [H_{\omega,a}^M, \chi_x] R_{\omega,a,M} \chi_y \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_z, \quad (5.16)$$

with

$$\mathcal{S}_1 := \{\mathbb{Z} \times \{0\}\} \times \{(\mathbb{Z} \cap [-a-r_0, \infty)) \times \mathbb{Z}\} \times \{(\mathbb{Z} \cap (\infty, -r_0 - a - r]) \times (\mathbb{Z} \cap [-R, R])\}. \quad (5.17)$$

Notice that in (5.15), we have $|y - x| \leq 2$ and using Lemma A.5 asserting that the operator $R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} \chi_x$ is trace class independently of x , we obtain

$$\begin{aligned}
\|(5.15)\|_1 &\leq \sum_{x,y,z \in \mathcal{S}_1} \left\| R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} \chi_x \chi_y \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_z \right\|_1 \\
&\leq \sup_x \left\| R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} \chi_x \right\|_1 \sum_{x,y,z \in \mathcal{S}_1} \left\| \chi_x \chi_y \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_z \right\|_1 \\
&\leq \frac{C}{\eta^2 \tilde{\eta}} \sum_{\substack{(x_1, x_2) \in (\mathbb{Z} \cap [-a-r_0-2, \infty)) \times \{|x_2| < 1\} \\ (z_1, z_2) \in (\mathbb{Z} \cap (\infty, -r_0-a-r]) \times (\mathbb{Z} \cap [-R, R])}} e^{-c\tilde{\eta}(|z_1-x_1|+|z_2-x_2|)} \\
&\leq \frac{C_1}{\eta^2 \tilde{\eta}} e^{-c\tilde{\eta}r},
\end{aligned}$$

where $\eta = \text{dist}(z, \sigma(H_{\omega,a}^M))$ and $\tilde{\eta} = \text{dist}(z, \sigma(\tilde{H}_{\omega,a}^M))$. Since r is arbitrary and for R fixed, it follows that the trace vanishes. Next, we estimate (5.16). Let $\tilde{\chi}_x = 1$ on the support of $\nabla \chi_x$. Then, it suffices to use the decay of operator norms of $\chi_y \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_z$ and of $\tilde{\chi}_x [H_{\omega,a}, \chi_x] R_{\omega,a,M} \chi_y$. We thus obtain

$$\begin{aligned}
\|(5.16)\|_1 &\leq \sum_{x,y,z \in \mathcal{S}_1} \left\| R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} \tilde{\chi}_x [H_{\omega,a}^M, \chi_x] R_{\omega,a,M} \chi_y \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_z \right\|_1 \\
&\leq \sup_x \left\| R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} \tilde{\chi}_x \right\|_1 \sum_{x,y,z \in \mathcal{S}_1} \|\tilde{\chi}_x [H_{\omega,a}^M, \chi_x] R_{\omega,a,M} \chi_y\| \|\chi_y \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_z\| \\
&\leq \frac{c_1}{\eta^3 \tilde{\eta}} \sum_{x,y,z \in \mathcal{S}_1} e^{-c_2 \eta(|y_1-x_1|+|y_2-x_2|) - \tilde{c}_2 \tilde{\eta}(|z_1-y_1|+|z_2-y_2|)} \\
&\leq \frac{\tilde{c}_1(a+r_0)}{\eta^3 \tilde{\eta}} e^{-c\tilde{\eta}r}.
\end{aligned}$$

Taking $r \rightarrow \infty$, the trace of (5.16) vanishes and so does that of (5.9). A similar estimate holds for (5.10) so that we use (5.12), (5.13) and (5.14) to write

$$\begin{aligned}
(5.10) &= R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \left[\tilde{H}_{\omega,a}^M, \Lambda_2 \right] \tilde{R}_{\omega,a,M} \mathbf{1}_{|x_2| \leq R} \mathbf{1}_{x_1 \leq -r_0-r-a} \\
&= \sum_{x,y,z \in \mathcal{S}_1} R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_y \chi_x \left[\tilde{H}_{\omega,a}, \Lambda_2 \right] \tilde{R}_{\omega,a,M} \chi_z \quad (5.18)
\end{aligned}$$

$$\begin{aligned}
&+ \sum_{x,y,z \in \mathcal{S}_1} R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \left[\tilde{H}_{\omega,a}^M, \chi_y \right] \tilde{R}_{\omega,a,M} \chi_x \left[\tilde{H}_{\omega,a}^M, \Lambda_2 \right] \tilde{R}_{\omega,a,M} \chi_z, \quad (5.19)
\end{aligned}$$

where \mathcal{S}_1 is defined in (5.17). The trace class property holds for (5.18) from $R_{\omega,a}\mathcal{W}_{\omega,a}\tilde{R}_{\omega,a}\chi_y$ so that

$$\begin{aligned} \|(5.18)\|_1 &\leq \sum_{x,y,z \in \mathcal{S}_1} \left\| R_{\omega,a,M}\mathcal{W}_{\omega,a}\tilde{R}_{\omega,a,M}\chi_y\chi_x \left[\tilde{H}_{\omega,a}^M, \Lambda_2 \right] \tilde{R}_{\omega,a,M}\chi_z \right\|_1 \\ &\leq \sup_y \left\| R_{\omega,a,M}\mathcal{W}_{\omega,a}\tilde{R}_{\omega,a,M}\chi_y \right\|_1 \sum_{x,y,z \in \mathcal{S}_1} \left\| \chi_y\chi_x \left[\tilde{H}_{\omega,a}^M, \Lambda_2 \right] \tilde{R}_{\omega,a,M}\chi_z \right\| \\ &\leq \frac{c_2}{\tilde{\eta}^2\eta} \sum_{\substack{(x_1,x_2) \in (\mathbb{Z} \cap [-a-r_0-2,\infty)) \times \{|x_2| < 1\} \\ (z_1,z_2) \in (\mathbb{Z} \cap (\infty,-r_0-a-r]) \times (\mathbb{Z} \cap [-R,R])}} e^{-c\tilde{\eta}(|z_1-x_1|+|z_2-x_2|)} \\ &\leq \frac{\tilde{c}_2}{\eta^2\tilde{\eta}} e^{-c\tilde{\eta}r}. \end{aligned}$$

We have the analog procedure for (5.19) since we let $\tilde{\chi}_y = 1$ on $\text{supp } \nabla\chi_y$ and thus

$$\begin{aligned} \|(5.19)\|_1 &\leq \sum_{x,y,z \in \mathcal{S}_1} \left\| R_{\omega,a,M}\mathcal{W}_{\omega,a}\tilde{R}_{\omega,a,M}\tilde{\chi}_y \left[\tilde{H}_{\omega,a}^M, \chi_y \right] \tilde{R}_{\omega,a,M}\chi_x \left[\tilde{H}_{\omega,a}^M, \Lambda_2 \right] \tilde{R}_{\omega,a,M}\chi_z \right\|_1 \\ &\leq \sup_y \left\| R_{\omega,a,M}\mathcal{W}_{\omega,a}\tilde{R}_{\omega,a,M}\tilde{\chi}_y \right\|_1 \sum_{x,y,z \in \mathcal{S}_1} \left\| \chi_y \left[\tilde{H}_{\omega,a}^M, \chi_y \right] \tilde{R}_{\omega,a,M}\chi_x \right\| \left\| \chi_x \left[\tilde{H}_{\omega,a}^M, \Lambda_2 \right] \tilde{R}_{\omega,a,M}\chi_z \right\| \\ &\leq \frac{c_3}{\tilde{\eta}^3\eta} \sum_{x,y,z \in \mathcal{S}_1} e^{-c\tilde{\eta}(|x_1-y_1|+|x_2-y_2|+|z_1-x_1|+|z_2-x_2|)} \\ &\leq \frac{\tilde{c}_3(a+r_0)}{\tilde{\eta}^3\eta} e^{-c\tilde{\eta}r}. \end{aligned}$$

Since r is arbitrary, it follows that the trace of (5.18) and (5.19) vanish. Next, we estimate the trace norm of (5.11). Since $[\mathcal{W}_{\omega,a}, \Lambda_2] = 2i\tilde{\mathcal{A}}_a^{(2)} \cdot \nabla \Lambda_2$, we have

$$\begin{aligned} (5.11) &= R_{\omega,a,M} [\mathcal{W}_{\omega,a}, \Lambda_2] \tilde{R}_{\omega,a,M} \mathbf{1}_{|x_2| \leq R} \mathbf{1}_{x_1 \leq -r_0-r-a} \\ &= \sum_{x,y,z \in \tilde{\mathcal{S}}_1} R_{\omega,a,M} [\mathcal{W}_{\omega,a}, \Lambda_2] \chi_y \tilde{R}_{\omega,a,M} \chi_z \\ &= \sum_{x,y,z \in \tilde{\mathcal{S}}_1} \left(R_{\omega,a,M} [\mathcal{W}_{\omega,a}, \Lambda_2] \tilde{R}_{\omega,a,M} \chi_y \chi_z + R_{\omega,a,M} [\mathcal{W}_{\omega,a}, \Lambda_2] \tilde{R}_{\omega,a,M} \tilde{\chi}_y \left[\tilde{H}_{\omega,a}^M, \chi_y \right] \tilde{R}_{\omega,a,M} \chi_z \right), \end{aligned} \tag{5.20}$$

where

$$\tilde{\mathcal{S}}_1 := \{(\mathbb{Z} \cap [-a-r_0, \infty)) \times \{0\}\} \times \{(\mathbb{Z} \cap (\infty, -r_0-a-r]) \times (\mathbb{Z} \cap [-R, R])\}. \tag{5.21}$$

Since y and z lie in disjoint supports, we notice that the l.h.s of (5.20) zero. It remains thus to deal with $R_{\omega,a,M} [\mathcal{W}_{\omega,a}, \Lambda_2] \tilde{R}_{\omega,a,M} \chi_y \left[\tilde{H}_{\omega,a}^M, \chi_y \right] \tilde{R}_{\omega,a,M} \chi_z$ so that

$$\begin{aligned} \|(5.11)\|_1 &\leq \sup_y \left\| R_{\omega,a,M} [\mathcal{W}_{\omega,a}, \Lambda_2] \tilde{R}_{\omega,a,M} \chi_y \right\|_1 \sum_{y,z \in \tilde{\mathcal{S}}_1} \left\| \tilde{\chi}_y \left[\tilde{H}_{\omega,a}^M, \chi_y \right] \tilde{R}_{\omega,a,M} \chi_z \right\| \\ &\leq \frac{c_4}{\eta\tilde{\eta}^2} \sum_{y,z \in \tilde{\mathcal{S}}_1} e^{-c\tilde{\eta}(|z_1-y_1|+|z_2-y_2|)} \\ &\leq \frac{\tilde{c}_4}{\eta\tilde{\eta}^2} e^{-c\tilde{\eta}r}, \end{aligned}$$

which goes to 0 for r arbitrarily chosen. We deal now with the term (Π_R) in (5.2) that we treat exactly in the same way as (5.6). Indeed, following the previous steps, we have to check the trace norm of

$$R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \Lambda_1 \mathbf{1}_{|x_2|>R}, \quad (5.22)$$

and

$$R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} [\tilde{H}_{\omega,a}^M, \Lambda_2] \tilde{R}_{\omega,a,M} \Lambda_1 \mathbf{1}_{|x_2|>R}, \quad (5.23)$$

and

$$R_{\omega,a,M} [\mathcal{W}_{\omega,a}, \Lambda_2] \tilde{R}_{\omega,a,M}. \quad (5.24)$$

We write

$$\Lambda_1 \mathbf{1}_{|x_2|>R} = \sum_{(z_1, z_2) \in (\mathbb{Z}^- \times (\mathbb{Z} \cap [-R, R]^c))} \chi_z, \quad (5.25)$$

and we start with (5.22) that we express as

$$\begin{aligned} (5.22) &= R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] \chi_{|x_2| \leq 1} R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \Lambda_1 \mathbf{1}_{|x_2|>R} \\ &= \sum_{x,y,z \in \mathcal{S}_2} R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] \chi_x R_{\omega,a,M} \chi_y \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_z \\ &= \sum_{x,y,z \in \mathcal{S}_2} R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} \chi_x \chi_y \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_z \end{aligned} \quad (5.26)$$

$$+ \sum_{x,y,z \in \mathcal{S}_2} R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} [H_{\omega,a}^M, \chi_x] R_{\omega,a,M} \chi_y \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_z, \quad (5.27)$$

similarly to (5.9), where

$$\mathcal{S}_2 := \{\mathbb{Z} \times \{0\}\} \times \{(\mathbb{Z} \cap [-a - r_0, \infty)) \times \mathbb{Z}\} \times \{\mathbb{Z}^- \times (\mathbb{Z} \cap [-R, R]^c)\}. \quad (5.28)$$

To estimate the trace norm of (5.26) and (5.27), we follow (5.15) and (5.16) and we get

$$\begin{aligned} \|(5.26)\|_1 &\leq \sum_{x,y,z \in \mathcal{S}_2} \left\| R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} \chi_x \chi_y \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_z \right\|_1 \\ &\leq \frac{C}{\eta^2 \tilde{\eta}} \sum_{\substack{(x_1, x_2) \in (\mathbb{Z} \cap [-a - r_0 - 2, \infty)) \times \{|x_2| \leq 1\} \\ (z_1, z_2) \in \mathbb{Z}^- \times (\mathbb{Z} \cap [-R, R]^c)}} e^{-c\tilde{\eta}(|z_1 - x_1| + |z_2 - x_2|)} \\ &\leq \frac{C_1}{\eta^2 \tilde{\eta}} e^{c\tilde{\eta}(a+r_0)} e^{-c\tilde{\eta}R}, \end{aligned}$$

and

$$\begin{aligned} \|(5.27)\|_1 &\leq \sum_{x,y,z \in \mathcal{S}_2} \left\| R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] R_{\omega,a,M} [H_{\omega,a}^M, \chi_x] \tilde{\chi}_x R_{\omega,a,M} \chi_y \tilde{R}_{\omega,a,M} \chi_z \right\|_1 \\ &\leq \frac{c_1}{\eta^2 \tilde{\eta}} \sum_{x,y,z \in \mathcal{S}_2} e^{-c_2 \eta(|y_1 - x_1| + |y_2 - x_2|) - c_2 \tilde{\eta}(|z_1 - y_1| + |z_2 - y_2|)} \\ &\leq \frac{\tilde{c}_1(a + r_0)}{\eta^2 \tilde{\eta}} e^{-c\tilde{\eta}R}. \end{aligned}$$

Since R is arbitrary, we conclude that the traces of (5.26) and (5.27) vanish. Similarly to (5.10), we have

$$\begin{aligned} (5.23) &= R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \left[\tilde{H}_{\omega,a}^M, \Lambda_2 \right] \tilde{R}_{\omega,a,M} \Lambda_1 \mathbf{1}_{|x_2| > R} \\ &= \sum_{x,y,z \in \mathcal{S}_2} R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_y \chi_x \left[\tilde{H}_{\omega,a}^M, \Lambda_2 \right] \tilde{R}_{\omega,a,M} \chi_z \end{aligned} \quad (5.29)$$

$$+ \sum_{x,y,z \in \mathcal{S}_2} R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \left[\tilde{H}_{\omega,a}^M, \chi_y \right] \tilde{R}_{\omega,a,M} \chi_x \left[\tilde{H}_{\omega,a}^M, \Lambda_2 \right] \tilde{R}_{\omega,a,M} \chi_z, \quad (5.30)$$

where the set \mathcal{S}_2 is defined in (5.28). The trace norms of (5.29) and (5.30) are estimated similarly to that of (5.18) and (5.19), so that one has

$$\begin{aligned} \|(5.29)\|_1 &\leq \sum_{x,y,z \in \mathcal{S}_2} \left\| R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \chi_y \chi_x \left[\tilde{H}_{\omega,a}^M, \Lambda_2 \right] \tilde{R}_{\omega,a,M} \chi_z \right\|_1 \\ &\leq \frac{c_2}{\tilde{\eta}^2 \eta} \sum_{\substack{(x_1, x_2) \in (\mathbb{Z} \cap [-a-r_0-2, \infty)) \times \{|x_2| < 1\} \\ (z_1, z_2) \in \mathbb{Z}^- \times (\mathbb{Z} \cap [-R, R]^c)}} e^{-c\tilde{\eta}(|z_1-x_1|+|z_2-x_2|)} \\ &\leq \frac{\tilde{c}_2}{\eta^2 \tilde{\eta}} e^{c\tilde{\eta}(a+r_0)} e^{-c\tilde{\eta}R}, \end{aligned}$$

and

$$\begin{aligned} \|(5.30)\|_1 &\leq \sum_{x,y,z \in \mathcal{S}_1} \left\| R_{\omega,a,M} \mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M} \tilde{\chi}_y \left[\tilde{H}_{\omega,a}^M, \chi_y \right] \tilde{R}_{\omega,a,M} \chi_x \left[\tilde{H}_{\omega,a}^M, \Lambda_2 \right] \tilde{R}_{\omega,a,M} \chi_z \right\|_1 \\ &\leq \frac{c_3}{\tilde{\eta}^3 \eta} \sum_{x,y,z \in \mathcal{S}_1} e^{-c\tilde{\eta}(|x_1-y_1|+|x_2-y_2|+|z_1-x_1|+|z_2-x_2|)} \\ &\leq \frac{\tilde{c}_3(a+r_0)}{\tilde{\eta}^3 \eta} e^{-c\tilde{\eta}R}, \end{aligned}$$

with $\tilde{\chi}_y = 1$ on $\text{supp}(\nabla \chi_y)$. Next, we finish with (5.24) which is similar to (5.11) in the sense that

$$\begin{aligned} \|(5.24)\|_1 &\leq \sup \left\| R_{\omega,a,M} [\mathcal{W}_{\omega,a}, \Lambda_2] \tilde{R}_{\omega,a,M} \tilde{\chi}_y \right\|_1 \sum_{y,z \in \tilde{\mathcal{S}}_2} \left\| \tilde{\chi}_y \left[\tilde{H}_{\omega,a}^M, \chi_y \right] \tilde{R}_{\omega,a,M} \chi_z \right\|_1 \\ &\leq \frac{c_4}{\eta \tilde{\eta}^2} \sum_{y,z \in \tilde{\mathcal{S}}_1} e^{-c\tilde{\eta}(|z_1-y_1|+|z_2-y_2|)} \\ &\leq \frac{\tilde{c}_4}{\eta \tilde{\eta}^2} e^{-c\tilde{\eta}R}, \end{aligned}$$

where

$$\tilde{\mathcal{S}}_2 := \{(\mathbb{Z} \cap [-a-r_0, \infty)) \times \{0\}\} \times \{\mathbb{Z}^- \times (\mathbb{Z} \cap [-R, R]^c)\}. \quad (5.31)$$

Since R is arbitrarily chosen, we deduce that the traces of (5.23) and (5.24) are equal to zero.

• *Electric case.* For the reader's convenience, we sketch the main steps of the previous proof for the electric model. We split the operator $[g(H_{\omega,a}^E, \Lambda_2)] \Lambda_1$ in the x_2 -direction such that

$$[g(H_{\omega,a}^E, \Lambda_2)] \Lambda_1 = [g(H_{\omega,a}^E, \Lambda_2)] \Lambda_1 \mathbf{1}_{\{|x_2| \leq R\}} + [g(H_{\omega,a}^E, \Lambda_2)] \Lambda_1 \mathbf{1}_{\{|x_2| > R\}}, \quad (5.32)$$

For an arbitrary $R > 0$. In order to extract a compact part, we decompose the first r.h.s of (5.32) in the x_1 -direction for $r > 0$ arbitrary and we write it as

$$\left[g(H_{\omega,a}^E), \Lambda_2 \right] \mathbf{1}_{\{|x_2| \leq R\}} \mathbf{1}_{\{-r_0-a-r \leq x_1 \leq 0\}} + \left[g(H_{\omega,a}^E), \Lambda_2 \right] \mathbf{1}_{\{|x_2| \leq R\}} \mathbf{1}_{\{x_1 \leq -r_0-a-r\}}. \quad (5.33)$$

The trace of the l.h.s of (5.33) is zero, following the magnetic case. The assumptions on the electric potential U yields that there exists $r_0 > 0$ such that

$$U_a(x) \geq c_0, \quad \forall x_1 < -a - r_0, \quad (5.34)$$

where c_0 is chosen so that $c_0 > \sup \mathcal{I}$. The auxiliary operator that we consider is

$$\tilde{H}_{\omega,a}^E := H_{\omega,a}^E + c_0 \mathbf{1}_{x_1 \geq -r_0-a}. \quad (5.35)$$

In particular, $g(\tilde{H}_{\omega,a}^E) = 0$ since its spectrum is disjoint from \mathcal{I} . Otherwise, to treat the second terms in r.h.s of (5.33) and (5.32), we take advantage of the auxiliary operator $\tilde{H}_{\omega,a,M}$ defined in (5.35), as we did for (5.6) and (5.1), except that the first operator $\mathcal{W}_{\omega,a}$ is replaced by the operator W_a given by

$$W_a := \tilde{H}_{\omega,a}^E - H_{\omega,a}^E = c_0 \mathbf{1}_{x_1 \geq -r_0-a}. \quad (5.36)$$

We start by $\left[g(H_{\omega,a}^E), \Lambda_2 \right] \mathbf{1}_{\{|x_2| \leq R\}} \mathbf{1}_{\{x_1 \leq -r_0-a-r\}}$ that leads to check the trace norm of

$$R_{\omega,a,E} \left[H_{\omega,a}^E, \Lambda_2 \right] R_{\omega,a,E} W_a \tilde{R}_{\omega,a,E} \mathbf{1}_{\{|x_2| \leq R\}} \mathbf{1}_{\{x_1 \leq -r_0-a-r\}} \quad (5.37)$$

and

$$R_{\omega,a,E} W_a \tilde{R}_{\omega,a,E} \left[H_{\omega,a}^E, \Lambda_2 \right] R_{\omega,a,E} \mathbf{1}_{\{|x_2| \leq R\}} \mathbf{1}_{\{x_1 \leq -r_0-a-r\}}. \quad (5.38)$$

Once more, we use smooth decomposition of unity and we write

$$W_a = \sum_{\substack{y_1 \in \mathbb{Z} \cap (\infty, -r_0-a] \\ y_2 \in \mathbb{Z}}} \chi_z.$$

The term in (5.37) is treated in the same way as (5.9) where we have to estimate the trace norm of

$$\sum_{x,y,z \in \mathcal{S}_1} R_{\omega,a,E} \left[H_{\omega,a}^E, \Lambda_2 \right] R_{\omega,a,E} \chi_x \chi_y \tilde{R}_{\omega,a,E} \chi_z, \quad (5.39)$$

and

$$\sum_{x,y,z \in \mathcal{S}_1} R_{\omega,a,E} \left[H_{\omega,a}^E, \Lambda_2 \right] R_{\omega,a,E} \left[H_{\omega,a}^E, \chi_x \right] R_{\omega,a,E} \chi_y \tilde{R}_{\omega,a,E} \chi_z. \quad (5.40)$$

This follows from (5.15) and (5.16) where we use Lemma A.3 to obtain a decay of the kernel of $\mathcal{W}_{\omega,a} \tilde{R}_{\omega,a,M}$ instead of the Combes-Thomas estimate. In particular, (5.38) follows the same procedure as (5.10).

We now turn to the remaining term $\left[g(H_{\omega,a}^M) - g(\tilde{H}_{\omega,a}^M), \Lambda_2 \right] \Lambda_1 \mathbf{1}_{\{|x_2| > R\}}$ which is similar to (5.2). This gives analogous terms to (5.22) and (5.23) where once more, the decay of the kernel of the resolvent is replaced by that of $\mathcal{W}_{\omega,a} R_{\omega,a,M}$ thanks to Lemma A.3. After all, we conclude that the trace of $\left[g(H_{\omega,a}^M), \Lambda_2 \right] \Lambda_1$ vanishes.

5.1.2. *Trace class property.* In this section, we deal with the trace class property of the operators mentioned in Lemma 4.2.

The operator $g'(H_{\omega,a})[H_{\omega,a}, \Lambda_2] \Lambda_1$. The trace class property of this operator follows from the previous section where we have considered auxiliary operators $\tilde{H}_{\omega,a}$ to take advantage of the wall. Since we have $g'(\tilde{H}_{\omega,a}) = 0$, we should analyze the operator

$$(g'(H_{\omega,a}) - g'(\tilde{H}_{\omega,a}))[H_{\omega,a}, \Lambda_2] \Lambda_1$$

via the formula (4.2).

• *Magnetic case.* After computation and recalling that $\mathcal{W}_{\omega,a} = \tilde{H}_{\omega,a}^M - H_{\omega,a}^M$, we obtain six terms

$$\begin{aligned} & (\tilde{R}_{\omega,a,M}^3 - R_{\omega,a,M}^3) [H_{\omega,a}^M, \Lambda_2] \Lambda_1 \\ &= \left(\tilde{R}_{\omega,a,M}^2 \mathcal{W}_{\omega,a} R_{\omega,a,M} \tilde{R}_{\omega,a,M} + \tilde{R}_{\omega,a,M} \mathcal{W}_{\omega,a} R_{\omega,a,M}^2 \tilde{R}_{\omega,a,M} \right) [H_{\omega,a}^M, \Lambda_2] \Lambda_1 \\ &+ \left(\tilde{R}_{\omega,a,M} \mathcal{W}_{\omega,a} R_{\omega,a,M} \tilde{R}_{\omega,a,M}^2 + R_{\omega,a,M}^2 \tilde{R}_{\omega,a,M} \mathcal{W}_{\omega,a} R_{\omega,a,M} \right) [H_{\omega,a}^M, \Lambda_2] \Lambda_1 \\ &+ \left(R_{\omega,a,M} \tilde{R}_{\omega,a,M}^2 \mathcal{W}_{\omega,a} R_{\omega,a,M} + R_{\omega,a,M} \tilde{R}_{\omega,a,M} \mathcal{W}_{\omega,a} R_{\omega,a,M}^2 \right) [H_{\omega,a}^M, \Lambda_2] \Lambda_1. \end{aligned}$$

We shall treat one term and the others hold in quite similar way. For instance, we deal with $R_{\omega,a,M} \tilde{R}_{\omega,a,M} \mathcal{W}_{\omega,a} R_{\omega,a,M}^2 [H_{\omega,a}^M, \Lambda_2] \Lambda_1$ that we write as the sum of

$$R_{\omega,a,M} \tilde{R}_{\omega,a,M} \chi_y \mathcal{W}_{\omega,a} R_{\omega,a,M}^2 [H_{\omega,a}^M, \Lambda_2] \chi_x \quad (5.41)$$

over

$$\mathcal{D}_1 := \{(x_1, x_2) \in \mathbb{Z}^- \times (\mathbb{Z} \cap [-1, 1]), (y_1, y_2) \in (\mathbb{Z} \cap (\infty, -a - r_0]) \times \mathbb{Z}, u \in \mathbb{Z}^2\},$$

so that

$$\|(5.41)\|_1 \leq \left\| R_{\omega,a,M} \tilde{R}_{\omega,a,M} \chi_y \right\|_1 \left\| \chi_y \mathcal{W}_{\omega,a} R_{\omega,a,M} \chi_u \right\| \left\| \chi_u R_{\omega,a,M} [H_{\omega,a}^M, \Lambda_2] \chi_x \right\|. \quad (5.42)$$

Since $R_{\omega,a,M} \tilde{R}_{\omega,a,M} \chi_y$ is trace class independently of y and having in mind that $\mathcal{W}_{\omega,a}$ is a first order operator, we use Lemma A.3 to upper bound (5.42) by

$$c_1 |\operatorname{Im} z|^{-4} e^{-c_2 |\operatorname{Im} z| (|u-y| + |x-u|)}.$$

• *Electric case.* Once more, the same arguments work for the electric case subject to change $\mathcal{W}_{\omega,a}$ into W_a . If we consider the term $R_{\omega,a,E} \tilde{R}_{\omega,a,E}^2 W_a R_{\omega,a,E} [H_{\omega,a}^E, \Lambda_2] \Lambda_1$, we have to estimate the trace norm of the sum of

$$R_{\omega,a,E} \tilde{R}_{\omega,a,E}^2 \chi_y R_{\omega,a,E} [H_{\omega,a}^E, \Lambda_2] \chi_x \quad (5.43)$$

over

$$\mathcal{D}_2 := \{(x_1, x_2) \in \mathbb{Z}^- \times \{0\}, (y_1, y_2) \in (\mathbb{Z} \cap (\infty, -a - r_0]) \times \mathbb{Z}\}.$$

Thus the trace class property holds from $\tilde{R}_{\omega,a,E}^2 \chi_y$ while the summability of the sum comes out from the decay of $\chi_y R_{\omega,a,E} [H_{\omega,a}^E, \Lambda_2] \chi_x$ thanks to the Combes-Thomas estimate [CT]. In the sense that

$$\left\| R_{\omega,a,E} \tilde{R}_{\omega,a,E}^2 W_a R_{\omega,a,E} [H_{\omega,a}^E, \Lambda_2] \Lambda_1 \right\|_1 \leq C(a + r_0) |\operatorname{Im} z|^{-4}. \quad (5.44)$$

Now we establish the trace class property of operators depending on the time regularization $\Lambda_{1,a}^\omega(t)$.

The operator $[g(H_{\omega,a}), \Lambda_2, \Lambda_{1,a}^\omega(t)]$. In next analysis, we do not need to specify the case we deal with since the proof works for both electric and magnetic models. We subtract $[g(H_{\omega,a}), \Lambda_2] \Lambda_1$ which has zero trace by the previous analysis in section 5.1.1. Moreover, combining

$$\Lambda_{1,a}^\omega(t) - \Lambda_1 = i \int_0^t e^{isH_{\omega,a}} [H_{\omega,a}, \Lambda_1] e^{-isH_{\omega,a}} ds, \quad (5.45)$$

that we insert in (4.2) and the resolvent identity (4.4), we are left with the analysis of the trace norm of

$$R_{\omega,a}^2 [H_{\omega,a}, \Lambda_2] R_{\omega,a} e^{isH_{\omega,a}} [H_{\omega,a}, \Lambda_1] e^{-isH_{\omega,a}}, \quad (5.46)$$

and

$$R_{\omega,a} [H_{\omega,a}, \Lambda_2] R_{\omega,a}^2 e^{isH_{\omega,a}} [H_{\omega,a}, \Lambda_1] e^{-isH_{\omega,a}}. \quad (5.47)$$

These operators are localized in space in both directions x_1 and x_2 in the sense that each $[H_{\omega,a}, \Lambda_j]$ is localized on the support of Λ_j' because

$$[H_{\omega,a}^M, \Lambda_j] = -i(-i\nabla - A_0 - A_a - A_\omega) \cdot \nabla \Lambda_j - i \nabla \Lambda_j \cdot (-i\nabla - A_0 - A_a - A_\omega)$$

and

$$[H_{\omega,a}^E, \Lambda_j] = -i(-i\nabla - A_0) \cdot \nabla \Lambda_j - i \nabla \Lambda_j \cdot (-i\nabla - A_0).$$

To estimate the trace norm of (5.46), we decompose it with smooth characteristic functions and we rewrite

$$(5.46) = \sum_{\substack{(x_1, x_2) \in \mathbb{Z} \times \{0\} \\ (y_1, y_2) \in \{0\} \times \mathbb{Z}}} R_{\omega,a}^2 [H_{\omega,a}, \Lambda_2] \chi_x R_{\omega,a} e^{isH_{\omega,a}} [H_{\omega,a}, \Lambda_1] \chi_y e^{-isH_{\omega,a}}. \quad (5.48)$$

Since the operator $R_{\omega,a}^2 [H_{\omega,a}, \Lambda_2] \chi_x$ is trace class with

$$\|R_{\omega,a}^2 [H_{\omega,a}, \Lambda_2] \chi_x\|_1 \leq \frac{C}{|\operatorname{Im} z|^2},$$

and the operator norm of $\chi_x R_{\omega,a} e^{isH_{\omega,a}} [H_{\omega,a}, \Lambda_1] \chi_y$ is upper bounded by

$$e^{c_1 s} |\operatorname{Im} z|^{-1} e^{-c_2 |\operatorname{Im} z| (|x_1 - y_1| + |x_2 - y_2|)},$$

which follows from Lemma A.3, we obtain the sum (5.48) is finite and thus (5.46) is trace class. Moreover, there exist two constants c_1 and c_2 such that

$$\|(5.46)\|_1 \leq c_3 |\operatorname{Im} z|^{-3} e^{c_1 s}. \quad (5.49)$$

We turn to (5.47) that we expand in the following way

$$(5.47) = \sum_{\substack{u_1, y_2 \in \mathbb{Z}, x \in \mathbb{Z}^2 \\ u_2 = 0, y_1 = 0}} R_{\omega,a} [H_{\omega,a}, \Lambda_2] \chi_u R_{\omega,a} \chi_x e^{isH_{\omega,a}} R_{\omega,a} [H_{\omega,a}, \Lambda_1] \chi_y e^{-isH_{\omega,a}}.$$

In order to extract the decay in x_1 and y_2 , we use commutators to push χ_u to the left through the resolvent $R_{\omega,a}$. Let $\tilde{\chi}_u$ be a smooth function such that $\tilde{\chi}_u = 1$ on

$\text{supp} \nabla \chi_u$. Then we have

$$\begin{aligned}
\|(5.47)\|_1 &\leq \sum_{\substack{u_1, y_2 \in \mathbb{Z}, \\ u_2=0, y_1=0}} \left\| R_{\omega, a} [H_{\omega, a}, \Lambda_2] \chi_u R_{\omega, a} \chi_x e^{isH_{\omega, a}} R_{\omega, a} [H_{\omega, a}, \Lambda_1] \chi_y e^{-isH_{\omega, a}} \right\|_1 \\
&\leq \sum_{\substack{u_1, y_2 \in \mathbb{Z}, \\ u_2=0, y_1=0}} \left\| R_{\omega, a} [H_{\omega, a}, \Lambda_2] R_{\omega, a} \chi_u \chi_x e^{isH_{\omega, a}} R_{\omega, a} [H_{\omega, a}, \Lambda_1] \chi_y e^{-isH_{\omega, a}} \right\|_1 \\
&+ \sum_{\substack{u_1, y_2 \in \mathbb{Z}, \\ u_2=0, y_1=0}} \left\| R_{\omega, a} [H_{\omega, a}, \Lambda_2] R_{\omega, a} \tilde{\chi}_u [H_{\omega, a}, \chi_u] R_{\omega, a} \chi_x e^{isH_{\omega, a}} R_{\omega, a} [H_{\omega, a}, \Lambda_1] \chi_y e^{-isH_{\omega, a}} \right\|_1 \\
&\leq C_1 |\text{Im } z|^{-3} e^{c_1 s} \sum_{\substack{x_1, y_2 \in \mathbb{Z} \\ y_1=0 \\ x_2 \in \mathbb{Z} \cap [-2, 2]}} e^{-\tilde{c}_1 |\text{Im } z| (|x_1 - y_1| + |x_2 - y_2|)} \\
&+ C_2 |\text{Im } z|^{-4} e^{c_2 s} \sum_{\substack{u_1, y_2 \in \mathbb{Z}, x \in \mathbb{Z}^2 \\ u_2=0, y_1=0}} e^{-\tilde{c}_2 |\text{Im } z| (|x_1 - u_1| + |x_2 - u_2| + |x_1 - y_1| + |x_2 - y_2|)} \\
&\leq \tilde{C}_1 |\text{Im } z|^{-3} e^{c_1 s} + C_2 |\text{Im } z|^{-4} e^{c_2 s} \sum_{\substack{u_1, y_2 \in \mathbb{Z} \\ u_2=0, y_2=0}} e^{-\tilde{c}_2 |\text{Im } z| (|u_1 - y_1| + |u_2 - y_2|)},
\end{aligned}$$

where we have combined Combes-Thomas estimate [CT] and Lemma A.3 together with Lemma A.5. Hence, the summability follows and the operator (5.47) is finally trace class with

$$\|(5.47)\|_1 \leq C_3 e^{\tilde{c}_3 s} (|\text{Im } z|^{-3} + |\text{Im } z|^{-4}). \quad (5.50)$$

The operator $g'(H_{\omega, a}) [H_{\omega, a}, \Lambda_2] \Lambda_{1, a}^\omega(t)$. From (4.3), it follows that

$$\begin{aligned}
g'(H_{\omega, a}) [H_{\omega, a}, \Lambda_2] \Lambda_{1, a}^\omega(t) &= g'(H_{\omega, a}) [H_{\omega, a}, \Lambda_2] \Lambda_1 \\
&+ \frac{i}{\pi} \int_{\mathbb{R}^2} \int_0^t \bar{\partial} \tilde{G}(z) R_{\omega, a}^3(z) [H_{\omega, a}, \Lambda_2] e^{isH_{\omega, a}} [H_{\omega, a}, \Lambda_1] e^{-isH_{\omega, a}} ds du dv.
\end{aligned} \quad (5.51)$$

By the previous result on the operator $g'(H_{\omega, a}) [H_{\omega, a}, \Lambda_2] \Lambda_{1, a}^\omega(t)$, it suffices to treat (5.51) and to estimate the trace norm operator of

$$R_{\omega, a}^3 [H_{\omega, a}, \Lambda_2] e^{isH_{\omega, a}} [H_{\omega, a}, \Lambda_1] e^{-isH_{\omega, a}}, \quad (5.52)$$

that we write as

$$R_{\omega, a}^2 [H_{\omega, a}, \Lambda_2] R_{\omega, a} e^{isH_{\omega, a}} [H_{\omega, a}, \Lambda_1] e^{-isH_{\omega, a}} \quad (5.53)$$

$$- R_{\omega, a}^3 [H_{\omega, a}, [H_{\omega, a}, \Lambda_2]] R_{\omega, a} e^{isH_{\omega, a}} [H_{\omega, a}, \Lambda_1] e^{-isH_{\omega, a}}. \quad (5.54)$$

We thus have

$$\|(5.53)\|_1 \leq \sum_{\substack{x_1, y_2 \in \mathbb{Z}, \\ x_2=0, y_1=0}} \left\| R_{\omega, a}^2 [H_{\omega, a}, \Lambda_2] \chi_x \right\|_1 \left\| \chi_x R_{\omega, a} e^{isH_{\omega, a}} [H_{\omega, a}, \Lambda_1] \chi_y \right\|,$$

and

$$\|(5.54)\|_1 \leq \sum_{\substack{x_1, y_2 \in \mathbb{Z}, \\ x_2=0, y_1=0}} \left\| R_{\omega, a}^3 [H_{\omega, a}, [H_{\omega, a}, \Lambda_2]] \chi_x \right\|_1 \left\| \chi_x R_{\omega, a} e^{isH_{\omega, a}} [H_{\omega, a}, \Lambda_1] \chi_y \right\|.$$

The trace norms above are upper bounded by a constant c uniformly in x and the operator norms operators are bounded by

$$e^{c_1 s} |\operatorname{Im} z|^{-1} e^{-c_2 |\operatorname{Im} z| (|x_1 - y_1| + |x_2 - y_2|)}.$$

Then the trace class property holds.

Notice that although the operator $[g(H_{\omega,a}), \Lambda_2] \Lambda_{1,a}^\omega(t)$ is still trace class, there is no reason anymore for its trace to vanishes since Λ_2 does not commute with $\Lambda_{1,a}^\omega(t)$ as it is the case with Λ_1 .

5.2. Contributions of the Bulk quantities. We start by proving the zero contribution of the remainder term (4.9).

5.2.1. *Proof of Lemma 4.3.* For convenience we set

$$r_{\omega,a}^{(1)}(t) = R_{\omega,a}^2(z) [H_{\omega,a}, \Lambda_2] R_{\omega,a}(z) [H_{\omega,a}, \Lambda_{1,a}^\omega(t)] R_{\omega,a}(z), \quad (5.55)$$

$$r_{\omega,a}^{(2)}(t) = R_{\omega,a}(z) [H_{\omega,a}, \Lambda_2] R_{\omega,a}^2(z) [H_{\omega,a}, \Lambda_{1,a}^\omega(t)] R_{\omega,a}(z), \quad (5.56)$$

and

$$r_{\omega,a}^{(3)}(t) = R_{\omega,a}(z) [H_{\omega,a}, \Lambda_2] R_{\omega,a}(z) [H_{\omega,a}, \Lambda_{1,a}^\omega(t)] R_{\omega,a}^2(z), \quad (5.57)$$

that appear in (4.9). We first treat (5.56) and prove the convergence to the corresponds bulk quantity. Rewrite $r_{\omega,a}^{(2)}(t)$ as

$$[R_{\omega,a}, \Lambda_2] (H_\omega + \Theta) \langle x_2 \rangle^{2\nu} (\langle x_2 \rangle^{-2\nu} (H_\omega + \Theta)^{-2} \langle x_1 \rangle^{-2\nu}) \langle x_1 \rangle^{2\nu} (H_\omega + \Theta) [R_{\omega,a}, \Lambda_{1,a}^\omega(t)]. \quad (5.58)$$

We notice that the operators

$$[R_{\omega,a}, \Lambda_2] (H_\omega + \Theta) \langle x_2 \rangle^{2\nu} \quad \text{and} \quad \langle x_1 \rangle^{2\nu} (H_\omega + \Theta) [R_{\omega,a}, \Lambda_{1,a}^\omega(t)]$$

are uniformly bounded in a . As the middle operator $(\langle x_2 \rangle^{-2\nu} (H_\omega + \Theta)^{-2} \langle x_1 \rangle^{-2\nu})$ is trace class (see [BoGKS]), it follows from Lemma A.1 and Proposition A.2 that it suffices to prove the strong convergence of the left and right operators in (5.58) in $\mathcal{C}_c^\infty(\mathbb{R}^2)$. We use the identity $R_{\omega,a}^E - R_\omega^E = -R_{\omega,a}^E U_a R_\omega^E$ to write

$$[R_{\omega,a}^E - R_\omega^E, \Lambda_2] = \Lambda_2 R_{\omega,a}^E U_a R_\omega^E - R_{\omega,a}^E U_a R_\omega^E \Lambda_2. \quad (5.59)$$

Similarly, we have

$$R_{\omega,a}^M - R_\omega^M = -R_{\omega,a}^M \Gamma_{\omega,a} R_\omega^M,$$

for the magnetic model and thus

$$[R_{\omega,a}^M - R_\omega^M, \Lambda_2] = \Lambda_2 R_{\omega,a}^M \Gamma_{\omega,a} R_\omega^M - R_{\omega,a}^M \Gamma_{\omega,a} R_\omega^M \Lambda_2. \quad (5.60)$$

Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be such that $\operatorname{supp} \varphi \subset D_{r_1, r_2}$ where $D_{r_1, r_2} = [-r_1, r_1] \times [-r_2, r_2]$ for $r_1 < a$ and $r_2 > 0$. In particular, $\operatorname{supp}(\Lambda_2(H_\omega + \Theta) \langle x_2 \rangle^{2\nu} \varphi) \subset D_{r_1, r_2}$ and we have

$$\|R_{\omega,a}^M \Gamma_{\omega,a} R_\omega^M \Lambda_2(H_\omega + \Theta) \langle x_2 \rangle^{2\nu} \varphi\| \leq \frac{C}{|\operatorname{Im} z|^2} e^{-c|a-r_1|} \|\Lambda_2(H_\omega + \Theta) \langle x_2 \rangle^{2\nu} \varphi\|$$

and

$$\|\Lambda_2 R_{\omega,a}^M \Gamma_{\omega,a} R_\omega^M (H_\omega + \Theta) \langle x_2 \rangle^{2\nu} \varphi\| \leq \frac{C}{|\operatorname{Im} z|^2} e^{-c|a-r_1|} \|(H_\omega + \Theta) \langle x_2 \rangle^{2\nu} \varphi\|,$$

which converge to 0 as $a \rightarrow +\infty$. The electric case holds in a the same way.

Next we carry on the convergence of the right side of the operator in (5.58) and we write

$$[R_{\omega,a}, \Lambda_{1,a}^\omega(t)] - [R_\omega, \Lambda_1^\omega(t)] = [R_{\omega,a} - R_\omega, \Lambda_{1,a}^\omega(t)] + [R_\omega, \Lambda_{1,a}^\omega(t) - \Lambda_1^\omega(t)]. \quad (5.61)$$

We point out that the first term of the r.h.s of (5.61) is treated in the same spirit as (5.60) without time-dependence. In fact, one has

$$[R_{\omega,a} - R_\omega, \Lambda_{1,a}^\omega(t)] = [R_{\omega,a} - R_\omega, \Lambda_{1,a}^\omega(t) - \Lambda_1] + [R_{\omega,a} - R_\omega, \Lambda_1], \quad (5.62)$$

and the second term of the r.h.s of (5.62) looks like (5.60) where we have Λ_1 instead of Λ_2 . For the first term of (5.62), we take advantage of localisation in x_1 that the difference $\Lambda_{1,a}^\omega(t) - \Lambda_1$ gives us (see (5.45)) and the result holds similarly.

We come back to the second term in the r.h.s of (5.61), namely

$$\langle x_1 \rangle^{2\nu} (H_\omega + \Theta) [R_\omega, \Lambda_{1,a}^\omega(t) - \Lambda_1^\omega(t)],$$

that requires more works. We combine the commutator calculation and the first order resolvent identity to obtain

$$[R_\omega, \Lambda_1^\omega(t)] = -R_\omega e^{itH_\omega} [H_\omega, \Lambda_1] e^{-itH_\omega} R_\omega,$$

and

$$\begin{aligned} \Lambda_{1,a}^\omega(t) R_\omega &= \Lambda_{1,a}^\omega(t) R_{\omega,a} (1 + \Gamma_{\omega,a} R_\omega) \\ &= R_{\omega,a} \Lambda_{1,a}^\omega(t) (1 + \Gamma_{\omega,a} R_\omega) + e^{-itH_{\omega,a}} [R_{\omega,a}, \Lambda_1] e^{-itH_{\omega,a}} (1 + \Gamma_{\omega,a} R_\omega). \end{aligned}$$

Hence, one has

$$\begin{aligned} [R_\omega, \Lambda_{1,a}^\omega(t)] &= R_\omega \Lambda_{1,a}^\omega(t) - R_{\omega,a} \Lambda_{1,a}^\omega(t) (1 + \Gamma_{\omega,a} R_\omega) \\ &\quad - e^{-itH_{\omega,a}} [R_{\omega,a}, \Lambda_1] e^{-itH_{\omega,a}} (1 + \Gamma_{\omega,a} R_\omega), \end{aligned}$$

that we plug into $[R_\omega, \Lambda_{1,a}^\omega(t) - \Lambda_1^\omega(t)]$ to get

$$\begin{aligned} [R_\omega, \Lambda_{1,a}^\omega(t) - \Lambda_1^\omega(t)] &= (R_\omega - R_{\omega,a}) \Lambda_{1,a}^\omega(t) - R_{\omega,a} \Lambda_{1,a}^\omega(t) \Gamma_{\omega,a} R_\omega \\ &\quad - e^{-itH_{\omega,a}} [R_{\omega,a}, \Lambda_1] e^{-itH_{\omega,a}} (1 + \Gamma_{\omega,a} R_\omega) - e^{itH_\omega} [R_\omega, \Lambda_1] e^{-itH_\omega}. \end{aligned}$$

Hence, as $\Lambda_{1,a}^\omega(t) \rightarrow \Lambda_1^\omega(t)$ and $R_\omega \Gamma_{\omega,a} \rightarrow 0$ strongly, by Lemma B.1, the strong convergence to 0 as $a \rightarrow \infty$ follows.

Now, we deal with (5.55) and push one resolvent from the left through the commutator $[H_{\omega,a}, \Lambda_1]$, so that

$$r_{\omega,a}^{(1)}(t) = R_{\omega,a} [R_{\omega,a}, \Lambda_2] R_{\omega,a}^2 [H_{\omega,a}, \Lambda_{1,a}^\omega(t)] R_{\omega,a} \quad (5.63)$$

$$- R_{\omega,a}^2 [H_{\omega,a}, [H_{\omega,a}, \Lambda_2]] R_{\omega,a}^2 [H_{\omega,a}, \Lambda_{1,a}^\omega(t)] R_{\omega,a}. \quad (5.64)$$

The first term (5.63) fit exactly to (5.55). Proceeding as in (5.58) we get

$$\begin{aligned} (5.64) &= -R_{\omega,a} [R_{\omega,a}, [H_{\omega,a}, \Lambda_2]] (H_\omega + \Theta) \langle x_2 \rangle^{2\nu} (\langle x_2 \rangle^{-2\nu} (H_\omega + \Theta)^{-2} \langle x_1 \rangle^{-2\nu}) \\ &\quad \langle x_1 \rangle^{2\nu} (H_\omega + \Theta) [R_{\omega,a}, \Lambda_{1,a}^\omega(t)]. \end{aligned}$$

Once more, the middle operator $\langle x_2 \rangle^{-2\nu} (H_\omega + \Theta)^{-2} \langle x_1 \rangle^{-2\nu}$ is trace class [BoGKS]. By Lemma A.1 and Proposition A.2 together with Lemma B.1 and the fact that the right operator above $\langle x_1 \rangle^{2\nu} (H_\omega + \Theta) [R_{\omega,a}, \Lambda_{1,a}^\omega(t)]$ is previously treated in (5.60), we only need to prove the strong convergence of the operator

$$[R_{\omega,a}, [H_{\omega,a}, \Lambda_2]] (H_\omega + \Theta) \langle x_2 \rangle^{2\nu},$$

which is uniformly bounded in a . We compute the difference

$$[R_{\omega,a}, [H_{\omega,a}, \Lambda_2]] - [R_{\omega}, [H_{\omega}, \Lambda_2]] = [R_{\omega,a} - R_{\omega}, [H_{\omega,a}, \Lambda_2]] + [R_{\omega}, [\Gamma_{\omega,a}, \Lambda_2]], \quad (5.65)$$

and we let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ with support $D_{r_1, r_2} = [-r_1, r_1] \times [-r_2, r_2]$ for $r_1 < a$ and $r_2 > 0$. Then the supports of $[H_{\omega,a}, \Lambda_2] (H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi$ and $(H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi$ are both contained in D_{r_1, r_2} . Thus we estimate the operator norm of

$$\| [R_{\omega,a} - R_{\omega}, [H_{\omega,a}, \Lambda_2]] (H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi \| \quad (5.66)$$

so that

$$\begin{aligned} (5.66) &\leq \| R_{\omega} \Gamma_{\omega,a} R_{\omega,a} [H_{\omega,a}, \Lambda_2] (H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi \| \\ &\quad + \| [H_{\omega,a}, \Lambda_2] R_{\omega,a} \Gamma_{\omega,a} R_{\omega} (H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi \| \\ &\leq \| R_{\omega} \| \| \Gamma_{\omega,a} R_{\omega,a} [H_{\omega,a}, \Lambda_2] (H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi \| \\ &\quad + \| [H_{\omega,a}, \Lambda_2] R_{\omega,a} \| \| \Gamma_{\omega,a} R_{\omega} (H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi \| \\ &\leq C \left(|\operatorname{Im} z|^{-2} e^{-\tilde{c}_1 |\operatorname{Im} z| |a-r_1|} + |\operatorname{Im} z|^{-3/2} e^{-\tilde{c}_2 |\operatorname{Im} z| |a-r_1|} \right) \| (H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi \|, \end{aligned}$$

which converges to 0 as $a \rightarrow \infty$. Consider now the remaining term $[R_{\omega}, [\Gamma_{\omega,a}, \Lambda_2]]$ of the r.h.s of (5.65). We have

$$[R_{\omega}, [\Gamma_{\omega,a}, \Lambda_2]] = R_{\omega} [\Gamma_{\omega,a}, \Lambda_2] - \Gamma_{\omega,a} \Lambda_2 R_{\omega} + \Lambda_2 \Gamma_{\omega,a} R_{\omega},$$

and control its operator norm in the following way

$$\| \Gamma_{\omega,a} \Lambda_2 R_{\omega} (H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi \| \leq c_3 |\operatorname{Im} z|^{-1} e^{-\tilde{c}_3 |\operatorname{Im} z| |a-r_1|} \| (H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi \|, \quad (5.67)$$

and

$$\| \Lambda_2 \Gamma_{\omega,a} R_{\omega} (H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi \| \leq c_4 |\operatorname{Im} z|^{-1} e^{-\tilde{c}_4 |\operatorname{Im} z| |a-r_1|} \| (H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi \|, \quad (5.68)$$

while $R_{\omega} [\Gamma_{\omega,a}, \Lambda_2] (H_{\omega} + \Theta) \langle x_2 \rangle^{2\nu} \varphi = 0$ since $r_1 < a$. We thus conclude that (5.67) and (5.68) converge to 0 as $a \rightarrow \infty$.

In a similar way, we can establish the strong convergences in a of (5.57) to the bulk corresponding operators such that $r_{\omega,a}^{(3)}(t) \rightarrow r_{\omega}^{(3)}(t)$ where we denote by $r_{\omega}^{(1)}(t)$, $r_{\omega}^{(2)}(t)$ and $r_{\omega}^{(3)}(t)$ the analogous remainders.

Next, we estimate the time average of $r_{\omega}^{(j)}(t)$ in the trace norm for $j = 1, 2, 3$. In the first step, we introduce smooth characteristic functions $\chi_{\{|x_j| \leq R\}}$ and $\chi_{\{|x_j| > R\}}$ inside $r_{\omega}^{(1)}(t)$ where $R = T^{1/2}$ and $j = 1, 2$.

We rewrite $r_{\omega}^{(1)}(t)$ as the sum

$$R_{\omega}^2 [H_{\omega}, \Lambda_2] (\chi_{\{|x_1| \leq R\}} + \chi_{\{|x_1| > R\}}) R_{\omega} [H_{\omega}, \Lambda_1^{\omega}(t)] R_{\omega}. \quad (5.69)$$

We consider the time average of the r.h.s of (5.69) whose trace norm is estimated as

$$\begin{aligned} & \frac{1}{T} \|R_\omega^2 [H_\omega, \Lambda_2] \chi_{\{|x_1| \leq R\}} R_\omega (e^{iT H_\omega} \Lambda_1 e^{-iT H_\omega} - \Lambda_1) R_\omega\|_1 \\ & \leq \frac{1}{T} \|R_\omega^2 [H_\omega, \Lambda_2] \chi_{\{|x_1| \leq R\}}\|_1 \|R_\omega (e^{iT H_\omega} \Lambda_1 e^{-iT H_\omega} - \Lambda_1) R_\omega\| \\ & \leq \frac{CR}{T} |\operatorname{Im} z|^{-4}, \end{aligned}$$

which goes to 0 as $T \rightarrow \infty$ and where we have used the fact that operator $R_\omega^2 [H_\omega, \Lambda_2] \chi_{\{|x_1| \leq R\}}$ belongs to \mathcal{T}_1 together with

$$\begin{aligned} \frac{1}{T} \int_0^T [H_\omega, \Lambda_1^\omega(t)] dt &= \frac{1}{T} \int_0^T e^{it H_\omega} [H_\omega, \Lambda_1] e^{-it H_\omega} dt \\ &= \frac{-i}{T} (e^{iT H_\omega} \Lambda_1 e^{-iT H_\omega} - \Lambda_1). \end{aligned}$$

Concerning the second term of the r.h.s of (5.69), we have

$$\begin{aligned} & \left\| R_\omega^2 [H_\omega, \Lambda_2] \left(\frac{1}{T} \int_0^T \chi_{\{|x_1| > R\}} R_\omega e^{it H_\omega} [H_\omega, \Lambda_1] e^{-it H_\omega} R_\omega dt \right) \right\|_1 \\ & \leq \sum_{x, y \in \mathcal{N}_1} \left\| \frac{1}{T} \int_0^T R_\omega^2 [H_\omega, \Lambda_2] \chi_x R_\omega e^{it H_\omega} \chi_y [H_\omega, \Lambda_1] e^{-it H_\omega} R_\omega dt \right\|_1 \\ & \leq \sum_{x, y \in \mathcal{N}_1} \|R_\omega^2 [H_\omega, \Lambda_2] \chi_x\|_1 \left(\frac{1}{T} \int_0^T \|\chi_x R_\omega e^{it H_\omega} \chi_y\| dt \right) \| [H_\omega, \Lambda_1] e^{-it H_\omega} R_\omega \| \\ & \leq \tilde{C} |\operatorname{Im} z|^{-4} T^5 e^{-c_1 |\operatorname{Im} z| R}, \end{aligned}$$

where

$$\mathcal{N}_1 = \{(\mathbb{Z} \cap [-R, R]^c \times \{0\}) \times (\{0\} \times \mathbb{Z})\}. \quad (5.70)$$

Here, we have used the decay of the kernel $\chi_x R_\omega e^{it H_\omega} \chi_y$. Since $R = T^{\frac{1}{2}}$, the trace thus vanishes as $T \rightarrow \infty$.

The result $r_\omega^{(2)}(t)$ and $r_\omega^{(3)}(t)$ follows in quite similar way. For the reader's convenience, we nevertheless reproduce the details for $r_\omega^{(2)}(t)$.

$$r_\omega^{(2)}(t) = R_\omega [H_\omega, \Lambda_2] R_\omega \chi_{\{|x_1| \leq R\}} R_\omega [H_\omega, \Lambda_1^\omega(t)] R_\omega \quad (5.71)$$

$$+ R_\omega [H_\omega, \Lambda_2] R_\omega \chi_{\{|x_1| < R\}} R_\omega [H_\omega, \Lambda_1^\omega(t)] R_\omega \quad (5.72)$$

$$\begin{aligned} \frac{1}{T} \int_0^T \|(5.71)\|_1 dt &\leq \frac{1}{T} \|R_\omega [H_\omega, \Lambda_2] R_\omega \chi_{\{|x_1| \leq R\}}\|_1 \|R_\omega (e^{iT H_\omega} \Lambda_1 e^{iT H_\omega} - \Lambda_1) R_\omega\| \\ &\leq \frac{cR}{T} |\operatorname{Im} z|^{-4}. \end{aligned}$$

$$\begin{aligned}
\frac{1}{T} \int_0^T \|(5.72)\|_1 dt &\leq \sum_{x,y \in \mathcal{N}_2} \frac{1}{T} \|R_\omega [H_\omega, \Lambda_2] R_\omega \chi_x R_\omega e^{itH_\omega} \chi_y [H_\omega, \Lambda_1] e^{-itH_\omega} R_\omega\|_1 dt \\
&\leq \sup_x \|R_\omega [H_\omega, \Lambda_2] R_\omega \chi_x\|_1 \sum_{x,y \in \mathcal{N}} \left(\frac{1}{T} \|\chi_x R_\omega e^{itH_\omega} \chi_y\| dt \right) \|[H_\omega, \Lambda_1] e^{-itH_\omega} R_\omega\| \\
&\leq c_1 T^5 |\operatorname{Im} z|^{-4} e^{-c_2 |\operatorname{Im} z| R},
\end{aligned}$$

where

$$\mathcal{N}_2 = \{(\mathbb{Z} \cap [-R, R]^c \times \mathbb{Z}) \times (\{0\} \times \mathbb{Z})\}. \quad (5.73)$$

To conclude, we take the function \tilde{G} of order 5 so that the limit (4.10) follows.

5.2.2. Proof of Lemma 4.4. It follows from the section 5.2.1 that the operator $[g(H_{\omega,a}), \Lambda_2] (\Lambda_{1,a}^\omega(t) - \Lambda_1)$ is trace class. Concerning the convergence in trace to $[g(H_\omega), \Lambda_2] (\Lambda_1^\omega(t) - \Lambda_1)$, we adopt the same techniques used for the remainder operator $\mathcal{R}_{\omega,a}(t)$ in section 5.2.1. We use (4.2) and we notice that is enough to analyze the operators

$$R_{\omega,a} [H_{\omega,a}, \Lambda_2] R_{\omega,a}^2 (\Lambda_{1,a}^\omega(t) - \Lambda_1) \quad (5.74)$$

and

$$R_{\omega,a}^2 [H_{\omega,a}, \Lambda_2] R_{\omega,a} (\Lambda_{1,a}^\omega(t) - \Lambda_1). \quad (5.75)$$

Once again, we introduce the operator $(H_\omega + \Theta)^2$ inside (5.74) and (5.75). We write

$$\begin{aligned}
(5.74) &= -[R_{\omega,a}, \Lambda_2] (H_\omega + \Theta) \langle x_2 \rangle^{2\nu} (\langle x_2 \rangle^{-2\nu} (H_\omega + \Theta)^2 \langle x_1 \rangle^{-2\nu}) \\
&\quad \langle x_1 \rangle^{2\nu} (H_\omega + \Theta) R_{\omega,a} (\Lambda_{1,a}^\omega(t) - \Lambda_1).
\end{aligned} \quad (5.76)$$

Since the operator $(\langle x_2 \rangle^{-2\nu} (H_\omega + \Theta)^2 \langle x_1 \rangle^{-2\nu})$ is trace class [BoGKS], it suffices thanks to Lemma A.1, to prove the strong convergence of

$$[R_{\omega,a}, \Lambda_2] (H_\omega + \Theta) \langle x_2 \rangle^{2\nu} \quad \text{and} \quad \langle x_1 \rangle^{2\nu} (H_\omega + \Theta) R_{\omega,a} (\Lambda_{1,a}^\omega(t) - \Lambda_1),$$

in $\mathcal{C}_c^\infty(\mathbb{R}^2)$ as they are bounded uniformly in a . We notice that the operator $[R_{\omega,a}, \Lambda_2] (H_\omega + \Theta) \langle x_2 \rangle^{2\nu}$ has already been treated in (5.55). We are now left with $\langle x_1 \rangle^{2\nu} (H_\omega + \Theta) R_{\omega,a} (\Lambda_{1,a}^\omega(t) - \Lambda_1)$ that we rewrite as

$$\langle x_1 \rangle^{2\nu} (H_\omega + \Theta) (R_{\omega,a} - R_\omega) (\Lambda_{1,a}^\omega(t) - \Lambda_1) + \langle x_1 \rangle^{2\nu} (H_\omega + \Theta) R_\omega (\Lambda_{1,a}^\omega(t) - \Lambda_1).$$

Since $\Lambda_{1,a}^\omega(t) \rightarrow \Lambda_1^\omega(t)$ strongly, the second term converges to zero. To see that $\langle x_1 \rangle^{2\nu} (H_\omega + \Theta) (R_{\omega,a} - R_\omega)$ converges strongly to 0, we use $R_{\omega,a} - R_\omega = -R_\omega \Gamma_{\omega,a} R_{\omega,a}$ and we let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ compactly supported in D_{r_1, r_2} as in section 5.2.1 with $r_1 < a$. Then

$$\begin{aligned}
\|\langle x_1 \rangle^{2\nu} (H_\omega + \Theta) R_\omega \Gamma_{\omega,a} R_{\omega,a} \varphi\| &\leq \langle r_1 \rangle^{2\nu} \|(H_\omega + \Theta) R_\omega\| \|\Gamma_{\omega,a} R_{\omega,a} \varphi\| \\
&\leq \frac{C_{r_1}}{|\operatorname{Im} z|^{1/2}} e^{-c |\operatorname{Im} z| |a - r_1|} \|\varphi\|,
\end{aligned}$$

which converges to 0 as $a \rightarrow \infty$.

Next we turn to (5.75) whose analysis will be similar to that of (5.55). We commute $R_{\omega,a}$ and $[H_{\omega,a}, \Lambda_2]$ to write

$$\begin{aligned} (5.55) &= R_{\omega,a}^2 [H_{\omega,a}, \Lambda_2] R_{\omega,a} (\Lambda_{1,a}^\omega(t) - \Lambda_1) \\ &= R_{\omega,a} [H_{\omega,a}, \Lambda_2] R_{\omega,a}^2 (\Lambda_{1,a}^\omega(t) - \Lambda_1) \end{aligned} \quad (5.77)$$

$$- R_{\omega,a}^2 [H_{\omega,a}, [H_{\omega,a}, \Lambda_2]] R_{\omega,a}^2 (\Lambda_{1,a}^\omega(t) - \Lambda_1). \quad (5.78)$$

Since the first term (5.77) fit exactly to (5.74), we only need to check (5.78). We have

$$\begin{aligned} (5.78) &= R_{\omega,a} [R_{\omega,a}, [H_{\omega,a}, \Lambda_2]] (H_\omega + \Theta) \langle x_2 \rangle^{2\nu} (\langle x_2 \rangle^{-2\nu} (H_\omega + \Theta)^{-2} \langle x_1 \rangle^{-2\nu}) \\ &\quad \langle x_1 \rangle^{2\nu} (H_\omega + \Theta) R_{\omega,a} (\Lambda_{1,a}^\omega(t) - \Lambda_1). \end{aligned}$$

We notice that the right operator $\langle x_1 \rangle^{2\nu} (H_\omega + \Theta) R_{\omega,a} (\Lambda_{1,a}^\omega(t) - \Lambda_1)$ corresponds to the right operator treated in (5.76), while the left one

$$R_{\omega,a} [R_{\omega,a}, [H_{\omega,a}, \Lambda_2]] (H_\omega + \Theta) \langle x_2 \rangle^{2\nu}$$

has been treated in (5.65).

5.2.3. Proof of Lemma 4.5. According to the spectral theorem and the assumption on g , we have

$$g(H_\omega) = \int g(E) dP_\omega^{(E)}(E) = - \int g'(E) P_\omega^{(E)} dE, \quad (5.79)$$

since $g(+\infty)P_\omega^{(+\infty)} - g(-\infty)P_\omega^{(-\infty)} = 0$. Thanks to (5.79) we work with the Fermi projection $P_\omega^{(E)}$ and we are left with the study of $[P_\omega^{(E)}, \Lambda_2] (\Lambda_1^\omega(t) - \Lambda_1)$.

In the first step, we show that the operator $[P_\omega^{(E)}, \Lambda_2] (\Lambda_1^\omega(t) - \Lambda_1)$ is trace class uniformly in t . Using the Duhamel expansion 5.45, it is enough to prove that the operator

$$[P_\omega^{(E)}, \Lambda_2] e^{isH_\omega} [H_\omega, \Lambda_1] e^{isH_\omega}, \quad (5.80)$$

is trace class for $0 \leq s \leq t$. Notice that

$$[P_\omega^{(E)}, \Lambda_2] = P_\omega^{(E)} [P_\omega^{(E)}, \Lambda_2] + [P_\omega^{(E)}, \Lambda_2] P_\omega^{(E)}. \quad (5.81)$$

We introduce $(H_\omega - \Theta + 1)R_\omega(1 - \Theta)$ inside (5.80) such that

$$(5.80) = P_\omega^{(E)} [P_\omega^{(E)}, \Lambda_2] (H_\omega + \Theta - 1) e^{isH_\omega} R_\omega(1 - \Theta) [H_\omega, \Lambda_1] e^{isH_\omega} \quad (5.82)$$

$$+ [P_\omega^{(E)}, \Lambda_2] P_\omega^{(E)} (H_\omega + \Theta - 1) e^{isH_\omega} R_\omega(1 - \Theta) [H_\omega, \Lambda_1] e^{isH_\omega}. \quad (5.83)$$

We start with the term (5.83) that we rewrote as

$$\begin{aligned} &[P_\omega^{(E)}, \Lambda_2] e^{|x_2|^\zeta} \left(e^{-|x_2|^\zeta} P_\omega^{(E)} (H_\omega + \Theta - 1) e^{-|x_1|^\zeta} \right) \\ &\quad e^{|x_1|^\zeta} e^{isH_\omega} R_\omega(1 - \Theta) [H_\omega, \Lambda_1] e^{isH_\omega}. \end{aligned} \quad (5.84)$$

Since the operator $e^{-|x_2|^\zeta} P_\omega^{(E)} (H_\omega + \Theta - 1) e^{-|x_1|^\zeta}$ is well localized in energy and space, it is trace class. Moreover, the left and right operators in 5.84 are bounded by Lemma A.3 and Lemma A.4.

We come back now to (5.82) and use that

$$\left[P_\omega^{(E)}, \Lambda_2 \right] (H_\omega + \Theta - 1) = \left[P_\omega^{(E)}(H_\omega + \Theta - 1), \Lambda_2 \right] - P_\omega^{(E)} [H_\omega, \Lambda_2],$$

to write

$$(5.82) = P_\omega^{(E)} \left[P_\omega^{(E)}(H_\omega + \Theta - 1), \Lambda_2 \right] e^{isH_\omega} R_\omega (1 - \Theta) [H_\omega, \Lambda_1] e^{isH_\omega} \quad (5.85)$$

$$- P_\omega^{(E)} [H_\omega, \Lambda_2] e^{isH_\omega} R_\omega (1 - \Theta) [H_\omega, \Lambda_1] e^{isH_\omega}. \quad (5.86)$$

We expand these terms (5.85) and (5.86) as the sums of

$$P_\omega^{(E)} \chi_x \left[P_\omega^{(E)}(H_\omega + \Theta - 1), \Lambda_2 \right] \chi_y e^{isH_\omega} R_\omega (1 - \Theta) [H_\omega, \Lambda_1] \chi_u e^{isH_\omega} \quad (5.87)$$

and

$$- P_\omega^{(E)}(H_\omega + \Theta - 1) \chi_x R_\omega (1 - \Theta) [H_\omega, \Lambda_2] \chi_y e^{isH_\omega} R_\omega (1 - \Theta) [H_\omega, \Lambda_1] \chi_u e^{isH_\omega} \quad (5.88)$$

respectively over $\mathbb{Z}^2 \times \{\mathbb{Z} \times (\mathbb{Z} \cap [-1, 1])\} \times \{(\mathbb{Z} \cap [-1, 1]) \times \mathbb{Z}\}$. Since

$$\sup_x \left\| P_\omega^{(E)} \chi_x \right\|_1 < \infty \quad \text{and} \quad \sup_x \left\| P_\omega^{(E)}(H_\omega + \Theta - 1) \chi_x \right\|_1 < \infty,$$

we use Lemma A.3 to obtain an exponential decay of the kernels

$$\chi_y e^{isH_\omega} R_\omega (1 - \Theta) [H_\omega, \Lambda_1] \chi_u \quad \text{and} \quad \chi_x R_\omega (1 - \Theta) [H_\omega, \Lambda_2] \chi_y,$$

in operator norm to deduce the summability of (5.87) and (5.88). Therefore, the operator $\left[P_\omega^{(E)}, \Lambda_2 \right] (\Lambda_1^\omega(t) - \Lambda_1)$ is trace class. In the next step, we consider the decomposition

$$\left[P_\omega^{(E)}, \Lambda_2 \right] = \left[P_\omega^{(E)}, \Lambda_2 \right] P_\omega^{(E)\perp} + \left[P_\omega^{(E)}, \Lambda_2 \right] P_\omega^{(E)} = P_\omega^{(E)} \Lambda_2 P_\omega^{(E)\perp} - P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} \quad (5.89)$$

and we write

$$\begin{aligned} \left[P_\omega^{(E)}, \Lambda_2 \right] (\Lambda_1^\omega(t) - \Lambda_1) &= P_\omega^{(E)} \Lambda_2 P_\omega^{(E)\perp} (\Lambda_1^\omega(t) - \Lambda_1) \\ &\quad - P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} (\Lambda_1^\omega(t) - \Lambda_1). \end{aligned} \quad (5.90)$$

Both operators on the r.h.s of (5.90) are separately trace class. Hence, we can cycle the projections $P_\omega^{(E)}$ and $P_\omega^{(E)\perp}$ around the trace of (5.90). Setting

$$\Pi_E := P_\omega^{(E)} \Lambda_2 P_\omega^{(E)\perp} \Lambda_1 P_\omega^{(E)} - P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} \Lambda_1 P_\omega^{(E)\perp}, \quad (5.91)$$

and

$$\Pi_E(t) := P_\omega^{(E)} \Lambda_2 P_\omega^{(E)\perp} \Lambda_1^\omega(t) P_\omega^{(E)} - P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} \Lambda_1^\omega(t) P_\omega^{(E)\perp}, \quad (5.92)$$

one gets

$$\text{tr} (5.90) = \text{tr} \Pi_E(t) - \text{tr} \Pi_E. \quad (5.93)$$

We claim that the time-average of the trace of $\Pi_E(t)$ vanishes as T tends to ∞ . Indeed, we rewrite

$$\frac{1}{T} \int_0^T \Pi_E(t) \, dt = \frac{1}{T} \int_0^T (P_\omega^{(E)} \Lambda_2 P_\omega^{(E)\perp} \Lambda_1^\omega(t) P_\omega^{(E)} - P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} \Lambda_1^\omega(t) P_\omega^{(E)\perp}) \, dt, \quad (5.94)$$

as the sum of

$$\int_{\substack{\lambda > E \\ \mu \leq E}} \left(\frac{1}{T} \int_0^T e^{-it(\mu-\lambda)} dt \right) P_\omega^{(E)} \Lambda_2 P_\omega^{(E)\perp} dP_\lambda \Lambda_1 dP_\mu P_\omega^{(E)}, \quad (5.95)$$

and

$$\int_{\substack{\lambda \leq E \\ \mu > E}} \left(\frac{1}{T} \int_0^T e^{-it(\mu-\lambda)} dt \right) P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} dP_\lambda \Lambda_1 dP_\mu P_\omega^{(E)\perp}. \quad (5.96)$$

Since $\lambda \neq \mu$ and $|\frac{e^{ix}-1}{x}| \leq 1$, we have

$$\frac{1}{T} \int_0^T e^{-it(\mu-\lambda)} dt = \frac{e^{-iT(\mu-\lambda)} - 1}{-iT(\mu-\lambda)} \rightarrow 0,$$

when T tends to ∞ . Using the theorem of dominated convergence we complete the proof.

5.3. Bulk-Edge equality.

Proof of Lemma 4.6. We decompose the commutator within the bulk conductance (2.15) and we insert $-P_\omega^{(E)} \Lambda_2 \Lambda_1 P_\omega^{(E)} + P_\omega^{(E)} \Lambda_1 \Lambda_2 P_\omega^{(E)}$. Using that $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$, one obtains

$$\begin{aligned} \sigma_{\text{Hall}}(B, \omega, E) &= -i \operatorname{tr} \left[P_\omega^{(E)} \Lambda_2 P_\omega^{(E)}, P_\omega^{(E)} \Lambda_1 P_\omega^{(E)} \right] \\ &= i \operatorname{tr} (P_\omega^{(E)} \Lambda_2 P_\omega^{(E)\perp} \Lambda_1 P_\omega^{(E)} - P_\omega^{(E)} \Lambda_1 P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)}). \end{aligned} \quad (5.97)$$

Moreover, to see that

$$\operatorname{tr} (P_\omega^{(E)} \Lambda_1 P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)}) = \operatorname{tr} (P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} \Lambda_1 P_\omega^{(E)\perp}), \quad (5.98)$$

we apply Proposition A.2 and for instance we write

$$P_\omega^{(E)} \Lambda_1 P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} = \left[P_\omega^{(E)}, \Lambda_1 \right] P_\omega^{(E)\perp} \left[\Lambda_2, P_\omega^{(E)} \right], \quad (5.99)$$

which is seen to be trace class by cyclicity and Lemma A.4. The same argument works for $P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} \Lambda_1 P_\omega^{(E)\perp}$. We thus get

$$\operatorname{tr} (P_\omega^{(E)} \Lambda_1 P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)}) = \operatorname{tr} (P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} \Lambda_1 P_\omega^{(E)\perp}).$$

Recalling that

$$\Pi_E = P_\omega^{(E)} \Lambda_2 P_\omega^{(E)\perp} \Lambda_1 P_\omega^{(E)} - P_\omega^{(E)\perp} \Lambda_2 P_\omega^{(E)} \Lambda_1 P_\omega^{(E)\perp}, \quad (5.100)$$

one has $\sigma_{\text{Hall}}(E) = i \operatorname{tr} \Pi_E$ and (4.14) follows. \square

Theorem 4.1 is derived from the analysis done in the previous sections.

Proof of Theorem 4.1. Combining the fact that $\sigma_{\text{Hall}}(E) = i \operatorname{tr} \Pi_E$ and the constancy of Hall conductance σ_H in connexe intervals of localization, we conclude that

$$\sigma_{e,\omega}^{\text{reg}} = - \int g'(E) \sigma_{\text{Hall}}(E) dE = \sigma_{\text{Hall}}.$$

\square

APPENDIX A. TECHNICAL TOOLS

Lemma A.1. [Si] *Let $A_n \in \mathcal{B}$ such that $A_n \xrightarrow{s} A$ and let $B \in \mathcal{T}_p$ for $p > 0$. Then we have $\|A_n B - AB\|_p \rightarrow 0$.*

Proof. Since

$$\mathcal{T}_p = \overline{(\text{Finite rank operators})}_{\|\cdot\|_p}$$

there exists a finite rank operator P such that $\|(1 - P)B\|_p \leq \epsilon$ for a given $\epsilon > 0$. Write

$$\begin{aligned} \|(A_n - A)B\|_p &= \|(A_n - A)(B - PB + PB)\|_p \\ &\leq \|(A_n - A)P\| \|B\|_p + \|(A_n - A)\| \|(1 - P)B\|_p \\ &\leq \epsilon(\|A_n\| + \|A\| + \|B\|_p) \end{aligned}$$

where we have used that by strong convergence we have $(A_n - A)P \rightarrow 0$ and the result holds since ϵ is arbitrarily chosen. \square

Proposition A.2. [Si]

- (i) *Let $A_n \xrightarrow{s} A$ and B be a compact operator. Then $\|A_n B - AB\| \rightarrow 0$.*
- (ii) *Let $A, B \in \mathcal{B}$. If $AB, BA \in \mathcal{T}_1$ then $\text{tr } AB = \text{tr } BA$.*
- (iii) *Let $B \in \mathcal{B}$ and $A \in \mathcal{T}_1$. Then we have $\text{tr } AB = \text{tr } BA$.*
- (iv) *Let $A_n, B_n \in \mathcal{B}$ such that $A_n \xrightarrow{s} A$ and $B_n \xrightarrow{s} B$. Then $A_n B_n \xrightarrow{s} AB$.*

Next, we reproduce [CG, Lemma 3] that we adapt to obtain a decay of the kernel $\chi_x e^{-itH} R(z) [H, \Lambda_2] \chi_y$ in operator norm.

Lemma A.3. *Let χ_x and χ_y be two smooth functions. Let $R_A(z)$ be the resolvent of the operator $H(A) = (-i\nabla - A)^2$. Then there exist $c > 0$ and C_t such that*

$$\left\| \chi_x e^{-itH(A)} R_A(z) [H(A), \Lambda_2] \chi_y \right\| \leq \frac{C_t}{\eta} e^{-c\eta(|x_1 - y_1| + |x_2 - y_2|)} \quad (\text{A.1})$$

for all $z \notin \sigma(H(A))$ and $x, y \in \mathbb{R}^d$ and where $\eta = \text{dist}(z, \sigma(H))$.

Proof. We follow the same procedure used in [CG, Lemma 3]. We consider the vector potential $A = (0, \beta(x_1))$ and we let $\tilde{\chi}_j$ smooth functions with $\tilde{\chi}_j = 1$ on $\text{supp } \chi_j$ for $j = x, y$. We take $y_2 \in \text{supp } \Lambda_2'$ otherwise (A.1) is equal to zero.

We write $H(A) = (-i\nabla - A)^2 = \Pi_1^2 + \Pi_2^2$ where $\Pi_1 = p_1$ and $\Pi_2 = p_2 - \beta(x_1)$. Let us estimate the decay of $\chi_x e^{-itH(A)} R_A(z) [H, \Lambda_2] \chi_y$ for $t \in \mathbb{R}$. Notice that

$$\begin{aligned} [H(A), \Lambda_2] &= -i(-i\nabla - A) \cdot \nabla \Lambda_2 - i\nabla \Lambda_2 \cdot (-i\nabla - A) \\ &= -i\Pi_2 \Lambda_2' - i\Lambda_2' \Pi_2 \\ &= -\Lambda_2'' - 2i\Lambda_2' \Pi_2, \end{aligned}$$

and since for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we have

$$\left\| \chi_x e^{-itH(A)} R_A(z) \Pi_2 \chi_y \varphi \right\|^2 = \langle \chi_y \Pi_2 R_A(\bar{z}) e^{itH(A)} \chi_x^2 e^{-itH(A)} R_A(z) \Pi_2 \chi_y \varphi, \varphi \rangle,$$

it is enough to bound $\|\chi_y \Pi_2 R_A(\bar{z}) e^{itH(A)} \chi_x\|$. We write

$$\begin{aligned} \|\chi_y \Pi_2 R_A(\bar{z}) e^{itH(A)} \chi_x \varphi\|^2 &= \langle R_A(\bar{z}) e^{itH(A)} \chi_x \varphi, \Pi_2 \chi_y^2 \Pi_2 R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \rangle \\ &= \langle R_A(\bar{z}) e^{itH(A)} \chi_x \varphi, \tilde{\chi}_y (2(p_2 \chi_y) + \beta(x_1) \chi_y) \chi_y \Pi_2 R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \rangle \end{aligned} \quad (\text{A.2})$$

$$+ 2 \langle R_A(\bar{z}) e^{itH(A)} \chi_x \varphi, \chi_y^2 \Pi_2^2 R_A(\bar{z}) e^{itH(A)} \chi_1 \varphi \rangle, \quad (\text{A.3})$$

where we used

$$\Pi_2 \chi_y^2 \Pi_2 = (p_2 \chi_y^2) \Pi_2 + \chi_y^2 \Pi_2^2 = 2(p_2 \chi_y) \chi_y \Pi_2 + \chi_y^2 \beta(x_1) \Pi_2 + 2 \chi_y^2 \Pi_2^2$$

together with

$$p_2 \chi_y^2 = 2(p_2 \chi_y) \chi_y + \chi_y^2 p_2 = \chi_y^2 \Pi_2 + \tilde{\chi}_y (2(p_2 \chi_y) + \beta(x_1) \chi_y) \chi_y.$$

Similarly, we have

$$\Pi_1 \chi_y^2 \Pi_1 = (\Pi_1 \chi_y^2) \Pi_1 + \chi_y^2 \Pi_1^2$$

and

$$\Pi_1 \chi_y^2 = 2 \tilde{\chi}_y ((p_2 \chi_y) \chi_y + \chi_y^2 \Pi_1).$$

Hence

$$\begin{aligned} \|\chi_y \Pi_1 R_A(\bar{z}) e^{itH(A)} \chi_x \varphi\|^2 &= \langle R_A(\bar{z}) e^{itH(A)} \chi_x \varphi, \Pi_1 \chi_y^2 \Pi_1 R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \rangle \\ &= \langle R_A(\bar{z}) e^{itH(A)} \chi_x \varphi, \tilde{\chi}_y (2(p_2 \chi_y) \chi_y \Pi_2 R_A(\bar{z}) e^{itH(A)} \chi_x \varphi) \rangle \end{aligned} \quad (\text{A.4})$$

$$+ \langle R_A(\bar{z}) e^{itH(A)} \chi_x \varphi, \chi_y^2 \Pi_2^2 R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \rangle. \quad (\text{A.5})$$

We first estimate (A.2) so that

$$\begin{aligned} |(\text{A.2})| &\leq \|2(p_2 \chi_y) + \beta(x_1) \chi_y\|_\infty \left\| \tilde{\chi}_y R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \right\| \left\| \chi_x \Pi_2 R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \right\| \\ &\leq \frac{1}{2} \|2(p_2 \chi_y) + \beta(x_1) \chi_y\|_\infty^2 \left\| \tilde{\chi}_y R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \right\|^2 + \frac{1}{2} \left\| \chi_y \Pi_2 R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \right\|^2. \end{aligned}$$

In the same manner, one has

$$\begin{aligned} |(\text{A.4})| &\leq \|2(p_1 \chi_y)\|_\infty \left\| \tilde{\chi}_y R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \right\| \left\| \chi_y \Pi_1 R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \right\| \\ &\leq 2 \| (p_1 \chi_y) \|_\infty^2 \left\| \tilde{\chi}_y R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \right\|^2 + \frac{1}{2} \left\| \chi_y \Pi_1 R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \right\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|\chi_y \Pi_1 R_A(\bar{z}) e^{itH(A)} \chi_x \varphi\|^2 + \|\chi_y \Pi_2 R_A(\bar{z}) e^{itH(A)} \chi_x\|^2 \\ &\leq (\|2(p_2 \chi_y) + \beta(x_1) \chi_y\|_\infty^2 + 4 \| (p_1 \chi_y) \|_\infty^2) \left\| \tilde{\chi}_y R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \right\|^2 \\ &\quad + 2 |\langle R_A(\bar{z}) e^{itH(A)} \chi_x \varphi, \chi_y^2 (\Pi_1^2 + \Pi_2^2) R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \rangle| \\ &\leq (4 \| (p_1 \chi_y) \|_\infty^2 + 6 \| (p_2 \chi_y) \|_\infty^2 + 2 \| \beta(x_1) \chi_y \|_\infty^2) \left\| \tilde{\chi}_y R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \right\|^2 \\ &\quad + 2 \left\| \chi_y R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \right\| \left\| \chi_y \chi_x \varphi \right\| + 2 |\bar{z}| \left\| \chi_y R_A(\bar{z}) e^{itH(A)} \chi_x \varphi \right\|^2, \end{aligned} \quad (\text{A.6})$$

where we used that $(\Pi_1^2 + \Pi_2^2) R_A(\bar{z}) = I + \bar{z} R_A(\bar{z})$. Now since the function $\frac{e^{-itu}}{u-z}$ is analytic for $\text{Im } z \neq 0$ (then it is of Gevrey class), it follows from [GK3, BGK]

$$\left\| \chi_x e^{-itH(A)} R_A(z) \chi_y \right\| \leq \frac{e^{c_1 t}}{\eta} e^{-c_2 \eta |x-y|}, \quad (\text{A.7})$$

and the lemme holds. \square

The following lemma establishes the decay of the kernel operator of $[P_\omega^{(E)}, \Lambda_2]$.

Lemma A.4. *Assume (2.14). Then we have*

$$\left\| \chi_x [P_\omega^{(E)}, \Lambda_2] \chi_y \right\|_2 \leq C_{\omega, m, \zeta, \epsilon, B, E} e^{\epsilon |x|^\zeta} e^{-\frac{m}{2} |x_1 - y_1|^\zeta - \frac{m}{4} |x_2|^\zeta - \frac{m}{4} |y_2|^\zeta}, \quad (\text{A.8})$$

for all $x, y \in \mathbb{Z}^2$. Moreover, the operator $[P_\omega^{(E)}, \Lambda_1] P_\omega^{(E)\perp} [P_\omega^{(E)}, \Lambda_2]$ is trace class.

Proof. We recall the definition of the function Λ . We considered $\Lambda(s) = 1$ for $s \leq -\frac{1}{2}$ and $\Lambda(s) = 0$ for $s \geq \frac{1}{2}$ such that $\text{supp} \Lambda'_1 \subset (-\frac{1}{2}, \frac{1}{2})$.

We expand the commutator such that we have

$$\chi_x [P_\omega^{(E)}, \Lambda_2] \chi_y = \chi_x P_\omega^{(E)} \Lambda_2 \chi_y - \chi_x \Lambda_2 P_\omega^{(E)} \chi_y. \quad (\text{A.9})$$

If $x_2, y_2 \geq 1$ or $x_2, y_2 \leq -1$ then we have (A.9) = 0. Consider now the case $y_2 \leq -1, x_2 \geq 1$ or $y_2 \geq 1, x_2 \leq -1$. where we get (A.9) = $\pm \chi_x P_\omega^{(E)} \chi_y$. Thus the decay can be obtained by (2.14) and a use of

$$e^{-|x-y|^\zeta} \leq e^{-\frac{1}{2} |x_1 - y_1|^\zeta - \frac{1}{2} |x_2 - y_2|^\zeta},$$

and the fact that in the present case, we have $|x_2 - y_2|^\zeta = (|x_2| + |y_2|)^\zeta \geq \frac{1}{2} |x_1|^\zeta + \frac{1}{2} |x_2|^\zeta$. The case of $x_2 = 0$ or $y_2 = 0$ yields (A.8) since it follows from (2.14) for instance for $x_2 = 0$ that

$$\left\| \chi_x \Lambda_2 P_\omega^{(E)} \chi_y \right\|_2 \leq C_{\omega, m, \zeta, B, E} e^{\epsilon |x_1|^\zeta} e^{-m |x_1 - y_1|^\zeta - m |y_2|^\zeta}. \quad (\text{A.10})$$

Moreover, it follows from (A.8) that the operator $[P_\omega^{(E)}, \Lambda_1] P_\omega^{(E)\perp} [P_\omega^{(E)}, \Lambda_2]$ is trace class. \square

Lemma A.5. *Let R be the resolvent of the operator H . Then the operators*

$$\chi_x R^2 [H, \Lambda_j], \quad R^2 [H, \Lambda_j] \chi_x, \quad R [H, \Lambda_2] R \chi_x \in \mathcal{T}_1.$$

Proof. Let $M < \inf \sigma(H)$. We introduce the power resolvent $R^2(M)$ and we write

$$\chi_x R^2(z) [H, \Lambda_j] = \chi_x R^{\frac{3}{2}}(M) R^2(z) (H + M)^2 R^{\frac{1}{2}}(M) [H, \Lambda_j]. \quad (\text{A.11})$$

The trace class property follows from the fact that $R^2(z)(H+M)^2$ and $R^{\frac{1}{2}}(M) [H, \Lambda_j]$ are bounded and $\chi_x R^{\frac{3}{2}}(M)$ is trace class. In particular, $R^2(z) [H, \Lambda_j] \chi_x$ is also trace class since an operator T belongs to \mathcal{T}_1 if and only if $T^* \in \mathcal{T}_1$.

For the last operator, it suffices to see that

$$R [H, \Lambda_j] R \chi_x = [H, \Lambda_j] R^2 \chi_x - R [H, [H, \Lambda_j]] R^2 \chi_x, \quad (\text{A.12})$$

and since $[H, \Lambda_j] R^{\frac{1}{2}}$ is bounded as well as $R [H, [H, \Lambda_j]]$ and both $R^{\frac{3}{2}} \chi_x$ and $R^2 \chi_x$ are trace class then the operator $R [H, \Lambda_j] R \chi_x$ is trace class. \square

APPENDIX B. STRONG CONVERGENCE

Lemma B.1. *Let H_ω and $H_{\omega,a}$ be the operator defined in the sections 2 and 3 with corresponding resolvents R_ω and $R_{\omega,a}$. Then*

$$R_{\omega,a} \xrightarrow{s} R_\omega, \quad (\text{B.1})$$

as $a \rightarrow \infty$ and \mathbb{P} -a.e ω . In particular, one has

$$\Lambda_{1,a}^\omega(t) \xrightarrow{s} \Lambda_1^\omega(t) \quad \mathbb{P} - \text{a.e } \omega, \quad (\text{B.2})$$

for all $t \in \mathbb{R}$.

Proof. We have $R_{\omega,a} - R_\omega = -R_\omega \Gamma_{\omega,a} R_{\omega,a}$ and since R_ω is bounded, and $\Gamma_{\omega,a} R_{\omega,a}$ is uniformly bounded in a , it suffices to prove the strong convergence of $\Gamma_{\omega,a} R_{\omega,a}$ in $\mathcal{C}_0^\infty(\mathbb{R}^2)$. Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\text{supp} f = D_{r_1, r_2} = [-r_1, r_1] \times [-r_2, r_2]$ with $r_1 > a$ and $r_2 > 0$.

• *Electric case.* In this case, one has

$$\|\Gamma_{\omega,a}^E R_{\omega,a}^E f\| = \|U_a R_{\omega,a}^E f\| \leq c_1 |\text{Im } z|^{-1} e^{-\tilde{c}_1 |\text{Im } z| |a - r_1|} \quad (\text{B.3})$$

which goes to 0 as $a \rightarrow +\infty$ and where we have used Combes-Thomas estimate [CT, GK1].

• *Magnetic case.* Since the magnetic field \mathcal{B} is basically generated in the region $\mathcal{P}_a := (\infty, -a) \times \mathbb{R}$, it follows from [DGR1, Proposition 4.2] that we apply for this semi-plane \mathcal{P}_a , that the vector potential vanishes outside \mathcal{P}_a . This means that the operator

$$\Gamma_{\omega,a}^M = -2\mathcal{A}_a \cdot (-i\nabla - \mathcal{A}_0 - \mathcal{A}_\omega) + i \text{div } \mathcal{A}_a + |\mathcal{A}_a|^2$$

is supported on \mathcal{P}_a and the strong convergence of $\Gamma_{\omega,a}^M R_{\omega,a}^M f$ follows similarly to (B.3). The second point (B.2) is a consequence of (B.1), [RS]. \square

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UNIVERSITÉ DE TOULON, CNRS, CPT, UMR 7332, 83957 LA GARDE, FRANCE
E-mail address: `amal.taarabt@univ-tln.fr`